III Bsc VI Semester PCM

6.1 paper

Numerical Analysis

unit - 3.

From Newton Forward Interpolation Formula.

1. Newton Gregory forward interpolation formula

Let $y = f(x)$ be a function take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0+h, x_0+2h, \ldots$ of x. Let it is To find the value of y for $x = x_0 + ph$, where p is any real number.

Now we have $E^p f(x) = f(x+ph)$

$$\therefore y_x = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0$$

$$\Rightarrow y_x = \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} y_0$$

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

This is known as N.G.F.I.F.

2. N.G.B.I.F.

We know that $E^{-1} = 1 - \nabla$ i.e $E = (1-\nabla)^{-1}$

$$y_x = f(x_n+ph) = E^p f(x_n) = (1-\nabla)^{-p} y_n$$

$$\Rightarrow y_x = \left\{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\right\} y_n$$

$$\Rightarrow y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

This formula is known as N.G.B.I.F.

Problem

(1) Given that

x	1	2	3	4	5	6
$f(x)$	1	8	27	64	125	216

Estimate $f(7.5)$ using N.G.F.I.F.

Sol:- Forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	Δ^3	Δ^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

~~we shall take~~ Here $x_0 = 1$, $h = 1$, $y_0 = 1$

Here $x = 2.5$ & also $x = x_0 + ph$

$$\therefore p = \frac{x - x_0}{h} = \frac{2.5 - 1}{1} = 1.5$$

By N eq F.I.F, we have

$$f(2.5) = y_{2.5} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$f(2.5) = 1 + (1.5)(7) + \frac{(1.5)(0.5)}{2}(12) + \frac{(1.5)(0.5)(-0.5)}{6}(6)$$

$$= 1 + 10.5 + 4.5 - 0.375$$
$$= 15.625$$

(2) Given that $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60 = 0.8660$ find $\sin 52°$ using Newton-gregory forward interpolation formula

Sol:- The difference table is

$x°$	$10^4 y$	$10^4 \Delta$	$10^4 \Delta^2$	$10^4 \Delta^3$
45	7071			
		589		
50	7660		-57	
		532		-7
55	8192		-64	
		468		
60	8660			

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We Shall take

or

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

We Shall take $x_0 = 45$, $h = 5$, $y_0 = 0.7071$

Here $x = 52$ ∴ $p = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$

∴ N(e) FIF is

$$f(52) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\Rightarrow f(52) = 0.7071 + (1.4)(0.0589) + \frac{(1.4)(0.4)(-0.0057)}{2!} + \frac{(1.4)(0.4)(-0.6)}{3!}(-0.0007)$$

2×1

$= 0.7071 + 0.$

$\sin(52) = 0.7880$

(3) Ordinates $f(x)$ of a normal curves in terms of Standard deviation x are given as

x	1.00	1.02	1.04	1.06	1.08
$f(x)$	0.2420	0.2371	0.2323	0.2275	0.2227

find the ordinate for standard deviation $x = 1.025$ by using Newton Gregory forward interpolation formula.

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Soln: forward difference table is

x	$f(x)\times10^4$	$\Delta f(x)$	Δ^2	Δ^3	Δ^4
1.00	2420				
		-49			
1.02	2371		1		
		-48		-1	
1.04	2323		0		1
		-48		0	
1.06	2275		0		
		-48			
1.08	2227				

Here $x_0 = 1$, $h = 0.02$ & $y_0 = 2420$

given $x = 1.025$ $\therefore p = \frac{x - x_0}{h} = \frac{1.025 - 1}{0.02} = 1.25$

By Newton-Gregory forward interpolation formula we have

$$10^4 \times f(1.025) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$f(1.025) = 2420 + (1.25)(-49) + \frac{(1.25)(0.25)}{2}(1)$$
$$+ \frac{(1.25)(0.25)(-0.75)}{6}(-1) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(1)$$

$$= 2420 - 61.25 + 0.15625 + 0.039065 + 0.017098$$

$10^4 f(1.025) = 2358.9625$

$\therefore f(1.025) = 0.23589$

(4) find the number of Student from the following data Secured marks not more than 45

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	35	48	70	40	22

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Soln:- first we shall prepare the cumulative frequency table.

Marks < (x)	40	50	60	70	80
No. of Stds (y)	35	83	153	193	215

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$40 = x_0$	$35 = y_0$				
		48			
50	83		22		
		70		-52	
60	153		-30		64
		40		12	
70	193		-18		
		22			
80	215				

Now $x_0 = 40$, $x = 45$, $h = 10$

$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$

by Newton's of FIF

$$y_{45} = y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0$$

$$y_{45} = y_x = 35 + (0.5)(48) + \frac{(0.5)(-0.5)}{2}(22) + \frac{(0.5)(-0.5)(-1.5)}{6}(-52) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(64)$$

$$y_{45} = 35 + 24 - 2.75 - 3.25 - 2.5$$
$$= 59 - 8.5$$
$$= 50.5 \simeq 51 \text{ Students}$$

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(5) find the cubic polynomial which take the following values

x	0	1	2	3
$f(x)$	1	2	1	10

Soln:-

Now the difference table for x and $f(x)$ is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

We shall take, $x_0 = 0$ and $p = \frac{x - x_0}{h} = \frac{x-0}{1} = x$

By NGFIF,

$$f(x) = f(0) + p\Delta f(0) + \frac{p(p-1)}{2!}\Delta^2 f(0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(0)$$

$$\Rightarrow f(x) = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$\Rightarrow f(x) = 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$\Rightarrow f(x) = 2x^3 - 7x^3 + 6x + 1$$

problems on NGBIF

(1) Estimate $f(4.2)$ from the table

x	0	2	4	6
$f(x)$	2	10	66	218

Soln:- The difference table is

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\Delta^3 f(x)$
0	2			
		8		
2	10		48	
		56		48
4	66		96	
		152		
6	218			

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Let us take $x_n = 6$, $x = 4.2$ & $h = 2$

Now $p = \frac{x - x_n}{h} = \frac{4.2 - 6}{2} = -0.9$

Newton Gregory Backward interpolation formula is

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

$$y_{4.2} = f(4.2) = y_6 + (-0.9)\nabla y_n + \frac{(-0.9)(0.1)}{2}\nabla^2 y_6 + \frac{(-0.9)(0.1)(1.1)}{3!}\nabla^3 y_6$$

$$y_{4.2} = f(4.2) = 218 + (-0.9)(152) - (0.045)(96) - (0.0165)(48)$$

$$y_{4.2} = 218 - 136.8 - 4.32 - 0.792$$
$$= 76.088$$

(2) From the following table find the value of tan 17°

$\theta°$	0	4	8	12	16	20	24
$\tan\theta$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

Soln:- The difference table is

x	$f(x) \times 10^4$	∇	∇^2	∇^3	∇^4
0	0				
		699			
4	699		7		
		706		8	
8	1405		15		-3
		721		5	
12	2126		20		7
		741		12	
16	2867		32		-5
		773		7	
20	3640		39		
		812			
24	4452				

$p = \frac{17 - 24}{4} = -1.75$

Now By N.G. BIF is

$$10^4 y_{17} = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$10^4 y_{17} = 4452 + (-1.75)(812) + \frac{(-1.75)(-0.75)}{2}39 + \frac{(-1.75)(-0.75)(0.25)(7)}{6}$$
$$+ \frac{(-1.75)(-0.75)(0.25)(1.25)(-5)}{24}$$

$= 4452 - 14(2) + 25.59 + 0.3828 - 0.0854$

$10^4 y_{14} = 3056.887$

Thus $\tan 14^\circ = 0.3056$

Interpolation with unequal intervals

Lagrange's formula for unequal intervals :-

Let $y = f(x)$ be a function whose values are $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced.

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

This formula is known as Lagrange's interpolation formula for unequal intervals.

Ex:-

① The following table gives the normal weight of babies during first four months of life.

Age in Months	2	5	8	10	12
Weight in kgs	4.4	6.2	6.7	7.5	8.1

Estimate by Lagrange's method, the normal weight of a baby of 7 months old

Soln:- Let x = age of a baby (in months)

f(x) = weight (in kgs)

x	2	5	8	10	12
$f(x)$	4.4	6.2	6.7	7.5	8.7

By Lagrange's formula we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$f(x) = \frac{(x-5)(x-8)(x-10)(x-12)}{(2-5)(2-8)(2-10)(2-12)}(4.4) + \frac{(x-2)(x-8)(x-10)(x-12)}{(5-2)(5-8)(5-10)(5-12)}(6.2)$$

$$+ \frac{(x-2)(x-5)(x-10)(x-12)}{(8-2)(8-5)(8-10)(8-12)}(6.7) + \frac{(x-2)(x-5)(x-8)(x-12)}{(10-2)(10-5)(10-8)(10-12)}(7.5)$$

$$+ \frac{(x-2)(x-5)(x-8)(x-10)}{(12-2)(12-5)(12-8)(12-10)}(8.7)$$

$\boxed{x=7}$

$$f(7) = \frac{2(-1)(-3)(-5)}{(-3)(-6)(-8)(-10)}(4.4) + \frac{5(-1)(-3)(-5)}{3(-3)(-5)(-7)}(6.2) + \frac{5(2)(-3)(-5)}{(6)(3)(-2)(-4)}(6.7)$$

$$+ \frac{(5)(2)(-1)(-5)}{(8)(5)(2)(-2)}(7.5) + \frac{5(2)(-1)(-3)}{(10)(7)(4)(2)}(8.7)$$

$$f(7) = -0.09 + 1.48 + 6.98 - 2.34 + 0.47$$

$f(7) = \underline{6.5 \text{ kg}}$

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(2) Apply Lagrange's formula to find f(5) and f(6) given that $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(7)=128$ and Explain why the results different from those obtained by $f(x)=2^x$

Sol:-

Here $x_0=1,\ x_1=2\ \ x_2=3\ \ x_3=7$
& $f(x_0)=2\ \ f(x_1)=4\ \ f(x_2)=8\ \ f(x_3)=128$
By Lagrange's formula,

$$f(x)=\frac{(x-2)(x-3)(x-7)}{(1-2)(1-3)(1-7)}(2)+\frac{(x-1)(x-3)(x-7)}{(2-1)(2-3)(2-7)}(4)$$
$$+\frac{(x-1)(x-2)(x-7)}{(3-1)(3-2)(3-7)}8+\frac{(x-1)(x-2)(x-3)}{(7-1)(7-2)(7-3)}(128)$$

$$f(x)=\frac{(x-2)(x-3)(x-7)}{-12}(2)+\frac{(x-1)(x-3)(x-7)}{5}(4)$$
$$+\frac{(x-1)(x-2)(x-7)}{-8}(8)+\frac{(x-1)(x-2)(x-3)}{120}128$$

f(5) & f(6)

$$\therefore f(5)=\frac{(3)(2)(-2)}{-12}(2)+\frac{(4)(2)(-2)}{5}(4)+\frac{(4)(3)(-2)}{-8}(8)$$
$$+\frac{(4)(3)(2)}{120}(128)$$
$$=2-12.8+24+25.6$$
$$=38.8$$

$$\therefore f(6)=\frac{(4)(3)(-1)}{-12}(2)+\frac{(5)(3)(-1)}{5}(4)+\frac{(5)(4)(-1)}{-8}(8)$$
$$+\frac{(5)(4)(3)}{120}(128)$$
$$=2-12+20+64$$
$$=74$$

But actual values of f(5) and f(6) are 32 & 64. i.e $f(5)=2^5=32$ & $f(6)=2^6=64$

The difference in value of $f(5)$ and $f(6)$ are due to the assumption of $f(x)$ as a polynomial, when it is an exponential functio of the form 2^x

(3) Using Lagrange's formula, prove that

$$y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}[(y_3 - y_1) - (y_{-1} - y_{-3})]$$

Soln:-

Let $x = -3, -1, 1, 3$

$\therefore y = f(x) = y_{-3}, y_{-1}, y_1, y_3$

Then from Lagrange's formula, we have

$$y_x = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_{-3}) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_{-1})$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_1) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-3+1)(-3-1)(-3-3)}(y_{-3}) + \frac{(x+3)(x-1)(x-3)}{(-1+3)(-1-1)(-1-3)}(y_{-1})$$

$$+ \frac{(x+3)(x+1)(x-3)}{(1+3)(1+1)(1-3)}(y_1) + \frac{(x+3)(x+1)(x-1)}{(3+3)(3+1)(3-1)} y_3$$

$(x=0)$

$$\therefore y_0 = -\frac{3}{48}(y_{-3}) + \frac{9}{16}(y_{-1}) + \frac{9}{16} y_1 - \frac{3}{48}(y_{+3})$$

$$= -\frac{1}{16} y_{-3} + \frac{8+1}{16} y_{-1} + \frac{8+1}{16} y_1 - \frac{1}{16} y_{-3}$$

$$y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}[(y_3 - y_1) - (y_{-1} - y_{-3})]$$

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Newton's Divided Difference formula (Δ)

Let $y_0, y_1, \ldots y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$.

from the definition of divided difference,

$$f(x_0, x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x_0, x) \quad (1)$$

In general,

$$\therefore f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0)$$

$$+ \ldots + (x - x_0)(x - x_1) \ldots (x - x_n) \Delta^n f(x_0)$$

This is known as Newton's divided difference formula or Newton's general interpolation formula with divided difference.

Example:-

(1) Using Newton's divided difference formula, find the values of $f(8)$ & $f(15)$ from the following table

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Soln:- By Newton's divided difference formula, we have

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0)$$

$$+ \ldots\ldots$$

The divided difference table for the

data is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0=4$	48				
		$\frac{100-48}{5-4}=52$			
$x_1=5$	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
$x_2=7$	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
$x_3=10$	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
$x_4=11$	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
$x_5=13$	2028				

$\therefore \quad f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15)$
$\qquad + (x-4)(x-5)(x-7)(1)$

$f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)$

$x=8$

$f(8) = 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)$

$f(8) = 48 + 208 + 180 + 12 = 448$

iii) $f(15) = 48 + 572 + 1650 + 880 = 3150$

(2) find the polynomial of lowest degree which assumes the values 10, 4, 40, 424 and 620 at $x = -2, 1, 3, 7$ and 8 also find the value of the polynomial at $x=6$.

Sol^n: Here the arguments & Entries are

x	-2	1	3	7	8
$f(x)$	10	4	40	424	620

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-2	10				
		$\frac{4-10}{1+2}=-2$			
1	4		$\frac{18+2}{3+2}=4$		
		$\frac{40-4}{3-1}=18$		$\frac{13-4}{7+2}=1$	
3	40		$\frac{96-18}{7-1}=13$		0
		$\frac{424-40}{7-3}=96$		$\frac{20-13}{8-1}=1$	
7	424		$\frac{196-96}{8-3}=20$		
		$\frac{620-424}{8-7}=196$			
8	620				

By Newton's divided difference formula,

$$f(x)=f(x_0)+(x-x_0)\Delta f(x_0)+(x-x_0)(x-x_1)\Delta^2 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$f(x)=10+(x+2)(-2)+(x+2)(x-1)(4)+(x+2)(x-1)(x-3)(1)$$

$$f(x)=10-2x-4+4x^2+4x-8+x^3-2x^2-5x+6$$

$$f(x)=x^3+2x^2-3x+4$$

$$f(6)=216+72-18+4=\underline{274}$$

HW

(3) given $\log_{10}654=2.8156$, $\log_{10}658=2.8182$, $\log_{10}659=2.8189$ and $\log_{10}661=2.8202$ find by divided difference formula the value of $\log_{10}656$.

Numerical Differentiation

The process of Evaluating the derivative of a function at some particular value of the independent variable, when a set of given values of the function is known, is called numerical differentiation.

WKT $1+\Delta = e^{hD}$

$\Rightarrow hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 - - -$

$\Rightarrow f'(a) = \frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \frac{1}{4}\Delta^4 f(a) + ---\right]$

or

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ---\right] \quad -(1)$$

$||^y$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + --\right] \quad -(2)$$

The formula (1) & (2) used to Compute the first and Second derivatives of $y=f(x)$ at any value of x where y is specified.
Now we shall find a formula to find the derivatives using backward difference formula.

$(1-\nabla)^{-1} = e^{hD}$

$hD = \log(1-\nabla)^{-1} = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + --$

$\Rightarrow f'(a) = \frac{1}{h}\left[\nabla f(a) + \frac{1}{2}\nabla^2 f(a) + \frac{1}{3}\nabla^3 f(a) + \frac{1}{4}\nabla^4 f(a) + --\right]$

$||^y$

$f''(a) = \frac{1}{h^2}\left[\nabla^2 f(a) + \nabla^3 f(a) + \frac{11}{12}\nabla^4 f(a) + --\right]$

Example

(1) A function is specified by the following table

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.00	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

find y' & y'' at $x=1$.

Soln:-

Here $h=0.05$. To compute y' and y'' we take $x_0 = 1.00$.

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.00	1.00						
		0.0247					
1.05	1.0247		-0.0006				
		0.0241		0			
1.10	1.0488		-0.0006		0.0002		
		0.0235		0.0002		-0.0005	
1.15	1.0723		-0.0004		-0.0003		0.0009
		0.0231		-0.0001		0.0004	
1.20	1.0954		-0.0005		0.0001		
		0.0226		0			
1.25	1.1180		-0.00005				
		0.0221					
1.30	1.1401						

Now the formula is

$$y'_{x_0} = \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{0.05}\left[0.0247 - \frac{1}{2}(-0.0006) + \frac{1}{3}(0) - \frac{1}{4}(0.0002)\right.$$

$$+\frac{1}{5}(-0.0005)-\frac{1}{6}(0.0009)\Big]$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 20[0.0247+0.0003-0.00005-0.00015]$$

$$= 0.494$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{1}{(0.05)^2}\Big[-0.0006-0+\frac{11}{12}(0.0002)-\frac{5}{6}(-0.0005)$$

$$+\frac{137}{180}(0.0009)\Big]$$

$$= 400(-0.0006+0.0001833+0.0004166+0.000685)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 0.2739$$

(2) The following table gives the value of Sin θ for different value of θ

θ	0°	10°	20°	30°	40°
Sin θ	0.000	0.1736	0.3420	0.5000	0.6428

find the value of cos 10°.

Soln:- Here h = 10. To compute cos 10° in Lander find $\frac{d}{d\theta}(\sin\theta)$, we take $x_0 = 10°$

The difference table is

θ	Sin θ	Δ	Δ^2	Δ^3	Δ^4
0	0.000				
		0.1736			
10	0.1736		-0.0052		
		0.1684		-0.0052	
20	0.3420		-0.0104		0.0004
		0.1580		-0.0048	
30	0.5000		-0.0152		
		0.1428			
40	0.6428				

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Now formula is ($10° = \frac{\pi}{180}$ radians)

$$\left(\frac{d(\sin\theta)}{d\theta}\right)_{\theta=\frac{\pi}{18}} = \frac{18}{\pi}\left[0.1684 - \frac{1}{2}(-0.0104) + \frac{1}{5}(-0.0046)\right]$$

$$= \frac{18}{\pi}\left[0.1684 + 0.0052 - 0.0016\right]$$

$$= \frac{18}{\pi}(0.1720) = \underline{0.9850}$$

(3) find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=54$ from the table

x	50	51	52	53	54
y	3.6840	3.7084	3.7325	3.7563	3.7798

(find y' & y'')

Soln:- Here $h=1$. To compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ we take $x_n=54$.

The difference table is

x	y	∇y	$\nabla^2 y$	∇^3
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		
54	3.7798			

Now by backward formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla f(a) + \frac{1}{2}\nabla^2 f(a) + \frac{1}{3}\nabla^3 f(a) + \cdots\right]$$

$$y'_{54} = \frac{1}{1}\left[0.0235 + \frac{1}{2}(-0.0003)\right]$$

$$= 0.0232$$

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Again, $\left(\frac{d^2y}{dx^2}\right)_{x=54} = y'_{54} = \frac{1}{1^2}\left[\nabla^2 f(a) + \nabla^3 f(a) + \dots\right]$

$= 1\left[-0.0003\right]$

$y'_{54} = -0.0003$

HW

(4) A function $y = f(x)$ is specified by the following table

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $y'(1.6)$ and $y''(1.6)$ at $x = 1$

(5) find $f'(2.2)$ and $f''(2.2)$ given the following table

x	1.4	1.6	1.8	2.0	2.2
$f(x)$	4.0552	4.9530	6.0496	7.3891	9.0250

(6) Find $f'(10)$ from the following data

x	6	13	20	27	34
$f(x)$	-13	23	899	17315	35606

Soln:-

The difference table is

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
6	-13				
		36			
13	23		840		
		876		14700	
20	899		15540		-28365
		16416		-13665	
27	17315		1875		
		18291			
34	35606				

In above table 10 lies between 0 and 19 also it is near to the starting point
By using forward newton's formula

$$f'(a) = \frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \frac{1}{4}\Delta^4 f(a) - \dots\right]$$

Here $h = 7$,

$$f'(10) = \frac{1}{7}\left[36 - \frac{1}{2}(840) + \frac{1}{3}(14700) - \frac{1}{4}(-28365)\right]$$

$$= \frac{1}{7}\left[36 - 420 + 4900 + 7091.25\right]$$

$$= \underline{1658.17}$$

Numerical Integration

Numerical integration is a process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$. If the integrand is a function of a single-valued, then the numerical integration is called as quadrature.

Newton's – Cote's Quadrature formula (or) General Quadrature formula

Let $I = \int_a^b f(x)\,dx$, where $f(x)$ takes the values $f(x_0), f(x_0+h), f(x_0+2h) \dots$ or $f(x_0), f(x_1), f(x_2) \dots f(x_n)$ for $x = x_0, x_1, x_2, x_3 \dots x_n$

Divide the interval (a, b) into n subintervals of width h, so that,

$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h, \ldots\ x_n = x_0 + nh = b$

Now $I = \int_{x_0}^{x_0+nh} f(x)\,dx$

Let $x = x_0 + rh \Rightarrow dx = h\,dr$. Also when $x = x_0$, $r = 0$ and when $x = x_0 + nh$, $r = n$

$\therefore I = \int_0^n f(x_0 + rh)\,h\,dr$

Now by Newton's forward interpolation formula we have

$$I = h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\right]dr$$

Integrating term by term, we get

$$I = h\left[y_0 r + \frac{r^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \ldots\right]_0^n$$

Applying the limits from 0 to n

$$I = h\left[\left(y_0 n + \frac{n^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right) - (0)\right]$$

$$I = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^2}{3} - \frac{n}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^3}{4} - n^2 + n\right)\Delta^3 y_0 + \frac{n}{4!}\left(\frac{n^3}{5} - \frac{3n^2}{2} + \frac{11n}{3} - 3\right)\Delta^4 y_0 + \ldots\right]$$

This formula is known as Newton-Cote's Quadrature formula or a general Quadrature formula where $n = 1, 2, 3 \ldots$

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(1) Trapezoidal Rule

By taking $n=1$ in the general quadrature formula & neglecting terms containing $\Delta^2 y_0, \Delta^3 y_0$, ..., we get

$$I_1 = \int_{x_0}^{x_0+nh} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

WKT $\Delta y_0 = y_1 - y_0$

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[y_0 + \frac{y_1 - y_0}{2}\right]$$

$$= h\left[2y_0 + y_1 - y_0\right]$$

$$= h\left[y_0 + y_1\right]$$

$|||^{ly}$ $n=2$

$$I_2 = \int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}\left[y_1 + y_2\right]$$

⋮

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding these n integrals, we obtain

$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

This formula is known as Trapezoidal Rule for numerical integration.

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(2) Simpson's one third rule ($\frac{1}{3}$rd rule)
[n is Even]

put n=2 in general quadrature formula & neglect third order difference & higher terms we have.

$$I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2}\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

lll-

$$I_2 = \int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

(n is Even)

Adding these n integrals, we obtain

$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

This formula is known as Simpson's $\frac{1}{3}$rd rule

To apply this rule the given interval must be divided into Even number of equal subintervals

(3) Simpson's three-Eight rule & $\frac{3}{8}$th rule

By taking n=3 in G.Q.F & neglect the terms containing $\Delta^4 y_0, \Delta^5 y_0, \ldots$ we get

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$$I_1 = \int_{x_0}^{x_0+3h} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(9-\frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 + y_0)\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 + y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

|||ly

$$I_2 = \int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$\vdots$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

adding these integrals we get

$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})\right]$$

This formula is known as Simpson's $3/8$th rule.

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(4) Weddle's Rule

By taking $n=6$ in the G.Q.F & neglecting terms containing $\Delta^7 y_0, \Delta^8 y_0 \dots$ we get

$$I_1 = \int_{x_0}^{x_0+6h} f(x)\,dx = 6h\left[y_0 + 3(\Delta y_0) + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{1}{6}\cdot\frac{41}{140}(\Delta^6 y_0)\right]$$

Now
$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = y_4 + 4y_3 + 6y_2 - 4y_1 + y_0$$
$$\Delta^5 y_0 = y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0$$
$$\Delta^6 y_0 = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0$$

Also replacing $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$ (Approximately), we get

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Ily

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

adding these integrals we get

$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \dots\right]$$

i.e.
$$I_n = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[(y_0 + y_n) + 5(y_1 + y_5 + \dots + y_{n-5} + y_{n-1}) + (y_2 + y_4 + \dots + y_{n-2}) + 2(y_6 + y_{12} + \dots + y_{n-6}) + 6(y_3 + y_9 + \dots + y_{n-3})\right]$$

This formula is known as weddle's rule, if n is only multiple of 6 & 12.

Example:-

(1) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(1) Trapezoidal rule (2) Simpson's 1/3 rule
(3) Simpson's 3/8 rule (4) weddle's rule

Sol:-

Divide the interval (0,6) into 6 parts each of width h=1.

Let $f(x) = \frac{1}{1+x^2}$ then

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.027

(i) Trapezoidal rule, n=6

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= \underline{1.4108}$$

(ii) Simpson's 1/3rd rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= \underline{1.3662}$$

(iii) Simpson's $3/8^{th}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385) + 2\times 0.1\right]$$

$$= 1.35708$$

(iv) Weddle's rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$= \frac{3}{10}\left[1+5(0.5)+0.2+6(0.1)+0.0588 +5(0.0385)+0.027\right]$$

$$= 1.3734$$

(2) Calculate $\int_4^{5.2} \log x \, dx$ by using

(i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's $3/8^{th}$ rule (iv) Weddle's rule

given

x	4.0	4.2	4.4	4.6	4.8	5	5.2
$\log x$	1.38629	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

Sol:- (i) By Trapezoidal rule, $n=6$ & $h=0.2$

$$\int_4^{5.2} \log x \, dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.38629+1.64865)+2(1.43508+1.48160+1.52605+1.56861+1.60943)\right]$$

$$= 1.82764$$

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(ii) Simpson's 1/3 rule

$$\int_4^{5.2} \log x \, dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1.38629+1.64865)+4(1.43508+1.52605+1.60943)+2(1.48160+1.56861)\right]$$

$$= 1.82784$$

(iii) Simpson's 3/8th rule

$$\int_4^{5.2} \log x \, dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3\times 0.2}{8}\left[(1.38629+1.64865)+3(1.43508+1.48160+1.56861+1.60943)+2(1.52605)\right]$$

$$= 1.82784$$

(iv) Weddle's rule

$$\int_4^{5.2} \log x \, dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$= \frac{3\times 0.2}{10}\left[1.38629+5\times 1.43508+1.48160+6\times 1.52605+1.56861+5\times 1.60943+1.64865\right]$$

$$= 1.82784$$

(3) Evaluate $\int_0^1 e^x \, dx$ approximately in steps of 0.2 using by Trapezoidal rule

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Soln:- Let $y_x = e^x$ and $h = 0.2$

x	0	0.2	0.4	0.6	0.8	1.0
y_x	1	1.2114	1.4918	1.8221	2.2255	2.7183

By Trapezoidal rule

$$\int_0^1 e^x\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(1+2.7183)+2(1.2114+1.4918+1.8221+2.2255)\right]$$

$$= 1.72194.$$

(4) Using Simpson's $3/8^{th}$ rule obtain an approximate value of $\int_0^{0.3}(2x-x^2)^{1/2}\,dx$

Soln:- Let $f(x) = y = (2x-x^2)^{1/2}$ and take $n=6$

$$h = \frac{b-a}{n} = \frac{0.3-0}{6} = 0.05$$

Now,

x	0	0.05	0.10	0.15	0.20	0.25	0.30
y	0	0.3122	0.4359	0.5268	0.6000	0.6614	0.7141

By Simpson's $3/8^{th}$ rule we have

$$\int_0^{0.3}(2x-x^2)^{1/2}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3\times 0.05}{8}\left[(0+0.7141)+3(0.3122+0.4359+0.6000+0.6614)+2(0.5268)\right]$$

$$= 0.14617$$

(5) Find the value of log 2 from $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's $1/3$ rule by dividing the range into four equal parts.

Soln:- Here $a=0$, $b=1$, $n=4$

So that $h = \frac{b-a}{n} = \frac{1}{4} = 0.25$

x	0	0.25	0.5	0.75	1
y	0	0.06153	0.2222	0.39560	0.5000

The Simpson's 1/3 rule gives

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$= \frac{0.25}{3}\left[(0+0.5) + 4(0.06153 + 0.39560) + 2(0.2222)\right]$$

$$= \underline{0.23107}$$

H.W. (1) Evaluate $\int_0^6 \frac{dx}{1+x}$ by using

(i) Trapezoidal (ii) Simpson's 1/3 & 3/8th rule (iii) Weddle's rule.

(2) Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$ by using Simpson's 3/8th rule, dividing the interval into 3 equal parts Hence find an approximate value of $\log\sqrt{2}$

— 0 —

VI Sem III BSc

Numerical Analysis

Mathematics - 6.1.

Problems on factorial notation [Numerical analysis]

1. Express $x^4 - 12x^3 + 24x^2 - 30x + 9$ in factorial notation and find its successive differences.

Soln Let $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$

Then $f(x) = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} - \cancel{30}\,D x^{(1)} + E \longrightarrow ①$

Dividing $f(x)$ by synthetic division by $x, x-1, x-2, x-3$ & $x-4$ successively, we get

$x=0$	1	-12	24	-30	9
	0	0	0	0	0
	1	-12	24	-30	$\boxed{9 = E}$
$=1$	0	1	-11	13	
	1	-11	13	$\boxed{-17 = D}$	
$=2$	0	2	-18		
	1	-9	$\boxed{-5 = C}$		
$=3$	0	3			
	1	$\boxed{-6 = B}$			
$=4$	0				
	$1 = A$				

Then from ① $f(x) = x^{(4)} - 6x^{(3)} - 5x^{(2)} - 17x^{(1)} + 9$, is the factorial

Now, $\Delta f(x) = 4x^{(3)} - 6\cdot 3x^{(2)} - 5\cdot 2\cdot x^{(1)} - 17\cdot 1\cdot x^{(0)} + 0.$

$\therefore \Delta f(x) = 4x^{(3)} - 18x^{(2)} - 10x^{(1)} - 17$

$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} - 10.$

$\Delta^3 f(x) = 24x^{(1)} - 36.$

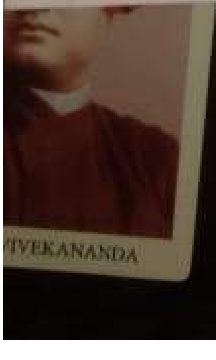
& $\Delta^4 f(x) = 24.$

2. Represent the polynomial $3x^3 + 3x^2 - 5x - 5$ in factorial symbol.

Soln Let $y = 3x^3 + 3x^2 - 5x - 5$

Then $y = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D \longrightarrow ①$

By synthetic division dividing y by $x, x-1, x-2$ & $x-3$



$$\begin{array}{r|cccc}
x=0 & 3 & 3 & -5 & -5 \\
 & 0 & 0 & 0 & 0 \\ \hline
 & 3 & 3 & -5 & \boxed{-5=D} \\
=1 & 0 & 3 & 6 & \\ \hline
 & 3 & 6 & \boxed{1=C} & \\
=2 & 0 & 6 & & \\ \hline
 & 3 & \boxed{12=B} & & \\
=3 & 0 & & & \\ \hline
 & 3=A & & &
\end{array}$$

$\therefore\ y = 3x^{(3)} + 12x^{(2)} + 1\cdot x^{(1)} - 5$ is the factorial notation.

Now, $\Delta y = 9x^{(2)} + 24x^{(1)} + 1$

$\Delta^2 y = 18x^{(1)} + 24$

& $\Delta^3 y = 18$

Definition: If $\Delta f(x) = g(x)$, then $f(x)$ is called the inverse forward difference of $g(x)$ and it is denoted by $f(x) = \Delta^{-1} g(x)$.

Thus, $\Delta f(x) = g(x) \Longleftrightarrow \Delta^{-1} g(x) = f(x)$. the Symbol Δ^{-1} is called inverse forward difference operator.

$$\therefore\ \Delta^{-1} x^{(n)} = \frac{x^{(n+1)}}{n+1} + K.$$

1. Find the polynomial whose first difference is $9x^2+11x+5$.

Soln Let $g(x) = 9x^2+11x+5$

$\therefore\ \Delta f(x) = g(x) = 9x^2+11x+5$.

Now, write $g(x)$ in factorial

$\therefore\ g(x) = Ax^{(2)} + Bx^{(1)} + C$

by synthetic division

$$\begin{array}{r|ccc}
x=0 & 9 & 11 & 5 \\
 & 0 & 0 & 0 \\ \hline
=1 & 9 & 11 & \boxed{5=C} \\
 & 0 & 9 & \\ \hline
=2 & 9 & \boxed{20=B} &
\end{array} \quad A=9$$

&

$\because x^{(0)} = 1$

$\therefore g(x) = 9x^{(2)} + 20x^{(1)} + 5.$

$\therefore \Delta f(x) = g(x) = 9x^{(2)} + 20x^{(1)} + 5x^{(0)}$

$$f(x) = \Delta^{-1} g(x)$$

$$= \Delta^{-1}\left[9x^{(2)} + 20x^{(1)} + 5x^{(0)}\right]$$

$$= 9\cdot\frac{x^{(2+1)}}{2+1} + 20\cdot\frac{x^{(1+1)}}{1+1} + 5\cdot\frac{x^{(0+1)}}{0+1} + K$$

$$= 3\cdot x^{(3)} + 10x^{(2)} + 5x^{(1)} + K$$

$$= 3\cdot x(x-1)(x-2) + 10\cdot x(x-1) + 5x + K$$

$\therefore f(x) = 3x^3 + x^2 + x + K$ is the required polynomial.

2. Obtain a function whose first difference is $3x^2 + 9x + 4$

Sol: Given, $\Delta f(x) = 3x^2 + 9x + 4$

$\therefore f(x) = \Delta^{-1}\left[3x^2 + 9x + 4\right] \rightarrow$ ①

Now, let $g(x) = 3x^2 + 9x + 4 = Ax^{(2)} + Bx^{(1)} + Cx^{(0)}$

then, by synthetic division, we have

$x=0$	3	9	4
	0	0	0
$=1$	3	9	$4 = C$
	0	3	
$=2$	3	$12 = B$	
	0		
	$3 = A$		

$\therefore g(x) = 3x^{(2)} + 12x^{(1)} + 4\cdot x^{(0)}$

from ① $\therefore f(x) = \Delta^{-1}\left[3\cdot x^{(2)} + 12\cdot x^{(1)} + 4x^{(0)}\right]$

$$= 3\cdot\frac{x^{(2+1)}}{2+1} + 12\cdot\frac{x^{(1+1)}}{1+1} + 4\cdot\frac{x^{(0+1)}}{0+1} + K$$

$$= x^{(3)} + 6x^{(2)} + 4x^{(1)} + K$$

$$= x(x-1)(x-2) + 6x(x-1) + 4x + K$$

$$= (x^2 - x)(x-2) + 6x^2 - 6x + 4x + K$$

$f(x) = x^3 + 3x^2 - x + K$ is the polynomial.

Method of Separation of Symbols.

We represent the function $f(x)$ by u_x then $f(x+nh)$ is denoted by u_{x+nh} and also, $f(0), f(1), f(2)$ etc are denoted by u_0, u_1, u_2 etc.

1. using Separation of Symbols prove

$$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^n u_{x-n}.$$

Sol: We have, $u_x - \Delta^n u_{x-n} = u_x - \Delta^n E^{-n} u_x$

$$= [1 - \Delta^n E^{-n}] u_x$$

$$= \left[1 - \frac{\Delta^n}{E^n}\right] u_x$$

$$= \left[\frac{E^n - \Delta^n}{E^n}\right] u_x$$

$$= \frac{1}{E^n}\left[\frac{E^n - \Delta^n}{I}\right] u_x$$ Where I is the identity operator, $If(x) = f(x)$

$$u_x - \Delta^n u_{x-n} = \frac{1}{E^n}\left[\frac{E^n - \Delta^n}{E - \Delta}\right] u_x$$ $\because E = I + \Delta$ (or) $E = 1 + \Delta$.

$$= \frac{1}{E^n}\left[E^{n-1} + E^{n-2}\Delta + E^{n-3}\Delta^2 + \cdots + E^0 \Delta^{n-1}\right] u_x$$

$$= [E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \Delta^3 E^{-4} + \cdots + \Delta^{n-1} E^{-n}] u_x$$

$$u_x - \Delta^n u_{x-n} = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \Delta^3 u_{x-4} + \cdots + \Delta^{n-1} u_{x-n}$$

$$\therefore u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$$

is the given result.

2. prove the identity

$u_0 - u_1 + u_2 - u_3 + u_4 - u_5 + \cdots\cdots =$

$\frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \cdots$

Soln: $LHS = u_0 - u_1 + u_2 - u_3 + \cdots\cdots$

$= u_0 - Eu_0 + E^2u_0 - E^3u_0 + \cdots$

$= [1 - E + E^2 - E^3 + \cdots\cdots]u_0$

$= [1+E]^{-1}u_0$

$= [1+1+\Delta]^{-1}u_0$

$= [2+\Delta]^{-1}u_0$

$= 2^{-1}\left[1+\frac{\Delta}{2}\right]^{-1}u_0$

$= \frac{1}{2}\left[1 - \frac{\Delta}{2} + \left(\frac{\Delta}{2}\right)^2 - \left(\frac{\Delta}{2}\right)^3 + \cdots\cdots\right]u_0$

$LHS = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \cdots\cdots$

$= RHS$

3. prove that $u_4 = u_3 + \Delta u_2 + \Delta^2 u_1 + \Delta^3 u_1$

Soln: Consider $RHS = u_3 + \Delta u_2 + \Delta^2 u_1 + \Delta^3 u_1$

$= E^2u_1 + \Delta E u_1 + \Delta^2 u_1 + \Delta^3 u_1$

$= [E^2 + \Delta E + \Delta^2 + \Delta^3]u_1$

$= [E^2 + \Delta E + \Delta^2(1+\Delta)]u_1$

$= [E^2 + \Delta E + \Delta^2 E]u_1 \quad \because 1+\Delta = E$

$= E[E + \Delta + \Delta^2]u_1$

$= E[E + \Delta(1+\Delta)]u_1$

$= E[E + \Delta E]u_1$

$RHS = E^2[1+\Delta]u_1 = E^2(E)u_1 = E^3u_1 = u_4 = LHS$

4. prove the identity $e^{x}\left[u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right]$

$$= u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \cdots\cdots$$

Soln consider $RHS = u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \cdots\cdots$

$$= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \cdots$$

$$= \left[1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \cdots\cdots\right]u_0$$

$$= e^{xE}u_0 = e^{x(1+\Delta)}u_0$$

$$= e^{x}\cdot e^{x\Delta}u_0$$

$$= e^{x}\left[1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \cdots\cdots\right]u_0$$

$$RHS. = e^{x}\left[u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right].$$

$$= LHS.$$

5. prove the identity $u_0 + u_1 + u_2 + \cdots + u_n =$

$$(n+1)C_1u_0 + (n+1)C_2\Delta u_0 + (n+1)C_3\Delta^2u_0 + \cdots + \Delta^n u_0$$

Soln $LHS = u_0 + u_1 + u_2 + u_3 + \cdots\cdots + u_n$

$$= [1 + E + E^2 + E^3 + \cdots + E^n]u_0$$

$$= \left[\frac{E^{n+1}-1}{E-1}\right]u_0 \qquad \because 1 + E + E^2 + \cdots + E^n \text{ is}$$

summation $S_n = \frac{r^n - 1}{r-1}$

$$= \left[\frac{(1+\Delta)^{n+1} - 1}{\Delta}\right]u_0$$

$$= \frac{1}{\Delta}\left[1 + (n+1)C_1\Delta + (n+1)C_2\Delta^2 + (n+1)C_3\Delta^3 + \cdots + \Delta^{n+1} - 1\right]$$

$$= [(n+1)C_1 + (n+1)C_2\Delta + (n+1)C_3\Delta^2 + \cdots + \Delta^n]u_0$$

$$LHS = (n+1)C_1u_0 + (n+1)C_2\Delta u_0 + (n+1)C_3\Delta^2u_0 + \cdots + \Delta^n u_0$$

$$= RHS$$

$u_4 = LHS$

Tan^{-1} R

cot^{-1} R

Interpolation

Interpolation is the technique of finding the value of a function $y=f(x)$ for any intermediate value of a the independent variable.

1. Newton - Gregory forward interpolation formula:

Let $y=f(x)$ be the function takes the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values $x_0, x_0+h, x_0+2h, \ldots x_0+nh$ of x where 'h' is interval of differencing. Let it is required to find the value of $f(x)$ for $x=x_0+ph$, where 'p' is any real number.

Now, $E^p f(x) = f(x+ph)$

Thus, $y_x = f(x) = f(x_0+ph)$

$$= E^p f(x_0)$$

$$= (1+\Delta)^p y_0 \qquad \because y_0 = f(x_0)$$

$$= \left[1 + \frac{p}{1!}\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right] y_0$$

$$\therefore y_x = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

This is known as Newton-Gregory forward interpolation formula which is used for interpolating the values near the beginning of a set of tabulated values.

2. Newton - Gregory backward interpolation formula.

Let $y=f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots$ corresponding to the arguments $x_0, x_0+h, x_0+2h, \ldots$ of x. Let it is required to find the value of $y=f(x)$ for ~~$x=x_0+ph$~~ $x=x_n+qh$, where 'q' is any real number.

Then, $y_x = f(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \cdots$

$\cdots$

This formula is known as Newton-Gregory backward interpolation formula.

1. From the following table find the number of students whose obtained < 45 marks.

May 2019

Marks	30-40	40-50	50-60	60-70	70-80
no of students	31	42	51	35	31

Soln: We shall prepare the cumulative frequency table.

Marks $<(x)$	40	50	60	70	80
No of stds (y)	31	73	124	159	190.

Now the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Here $x_0 = 40$, $x = 45$, $h = 10$ $\therefore x = x_0 + ph$ then

$p = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$.

By Newton-Gregory forward interpolation formula, we have

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y_x = 31 + 0.5\times 42 + \frac{0.5(0.5-1)}{2}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{6}\times(-25) +$$

$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}\times 37.$$

$$= 31 + 21 - 1.125 + \frac{0.5(-0.5)(-1.5)(-25)}{6} + \frac{0.5(-0.5)(-1.5)(-2.5)(37)}{24}$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$= 47.867 \approx$ 48 students

2. Find the number of students from the following data who secured marks not more than 45

Marks	30-40	40-50	50-60	60-70	70-80
no of students	35	48	70	40	22

Soln: First we shall prepare the cumulative frequency table for the data.

Marks < x:	40	50	60	70	80
no of stds y:	35	83	153	193	215

Now, the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 35$				
		48			
50	83		22		
		70		-52	
60	153		-30		64
		40		12	
70	193		-18		
		22			
80	215				

NOW, $x = 45$, $x_0 = 40$, $h = 10$, $p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$.

By NGB formula, we have

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$\therefore y_{45} = \cancel{40 + 0.5} \; 35 + 0.5 \times 48 + \frac{0.5(-0.5)}{2} \times 22 + \frac{0.5(-0.5)(-1.5)}{6}(-52)$$

$$+ \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 64$$

$$y_{45} = 35 + 24 - 2.75 - 3.25 - 2.5$$

$$y_{45} = 50.5 \approx \underline{51 \text{ students}}.$$

3. Given,

x :	1	2	3	4	5	6
$f(x)$:	1	8	27	64	125	216

Estimate $f(2.5)$.

Tan^{-1} R $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Cot^{-1}

Soln: Forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0=1$	$y_0=1$				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

Here $x=2.5$ lies in starting the data, by NGB formula

$$p=\frac{x-x_0}{h}, \quad x_0=1, \; h=1$$

$$=\frac{2.5-1}{1}=1.5$$

$$\therefore y_x=f(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\cdots$$

$$f(2.5)=1+1.5(7)+\frac{1.5(0.5)}{\cancel{2}}(\cancel{12})+\frac{1.5(0.5)(-0.5)}{\cancel{6}}(\cancel{6})$$

$$=1+10.5+4.5-0.375$$

$$\underline{f(2.5)=15.625}$$

4. Find the cubic polynomial which takes the following values

x:	0	1	2	3
$f(x)$:	1	2	1	10

Soln: First we shall construct the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$x_0 = 0$	$y_0 = 1$			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here $x_0 = 0$, $y_0 = 1$ then $p = \frac{x - x_0}{h} = \frac{x-0}{1} = x \quad \because h = 1$

We have by NGBF.

$$f(x) = y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$f(x) = 1 + x(1) + \frac{x(x-1)}{\cancel{2}}(\cancel{-2}) + \frac{x(x-1)(x-2)}{\cancel{6}}(\cancel{12})$$

$$= 1 + x - x^2 + x + 2x(x^2 - 3x + 2)$$

$$= 1 + 2x - x^2 + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 + 6x - 7x^2 + 1$$

$\underline{f(x) = 2x^3 - 7x^2 + 6x + 1}$ is the cubic polynomial.

5. Find $f(84)$ from the data.

x :	40	50	60	70	80	90
$f(x)$:	184	204	226	250	276	304

Sol: The difference table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
$x_n = 90$	$y_n = 304$			

Here $x_n = 90$, $h=10$, and $x=84$ is near the end of the table

then by NGBF we have, $q = \frac{x - x_n}{h} = \frac{84-90}{10} = -0.6$

Thus, $f(x) = y = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \cdots$

$$f(84) = 304 + (-0.6)\,28 + \frac{(-0.6)(-0.6+1)}{\cancel{2}}(\cancel{2}) + 0$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

$$\underline{f(84) = 287}$$

6. Estimate $f(4.2)$ from the table:

x:	0	2	4	6
$f(x)$:	2	10	66	218

Soln Difference table is as follow

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
0	2			
		8		
2	10		48	
		56		48
4	66		96	
		152		
$x_n = 6$	$y_n = 218$			

Here 4.2 is near the end of the data. Therefore we shall use NGBF to find $f(4.2)$

Here, $x_n = 6$, $x = 4.2$, $h = 2$, $q = \frac{x - x_n}{h} = \frac{4.2 - 6}{2} = -0.9$

$$f(4.2) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 218 + (-0.9)152 + \frac{(-0.9)(-0.9+1)}{2}.96 + \frac{(-0.9)(-0.9+1)(-0.9+2)}{6}.48$$

$$= 218 - 136.8 - 4.32 - 0.792$$

$$f(4.2) = 76.088$$

Interpolation with unequal intervals

1. Lagrange's interpolation formula for unequal intervals

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values of x are $x_0, x_1, x_2, \ldots x_n$ not neccessarily equally spaced

Then,
$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x-x_3)\cdots(x_0-x_n)}y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}y_1$$
$$+ \cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}y_n$$

This formula is known as lagrange's interpolation formula

Tan^{-1} R $[0, \pi]$ $(-\frac{\pi}{2}, \frac{\pi}{2})$ Cot^{-1}

1. Normal weight of the baby during the 1st eight months of life ~~at the age of 7 months using~~ is as follow: Age in months: 0 2 5 8

weight in kgs: 6 10 12 16

Estimate the weight of the baby at the age of 7 months using Lagrange's interpolation formula.

Soln Here age of the baby be 'x' $\therefore x_0=0,\ x_1=2,\ x_2=5,\ x_3=8$ and weight be 'y' $\therefore y_0=6,\ y_1=10,\ y_2=12,\ y_3=16$. It is required to find weight at $x=7$ month.

$$\therefore y_x = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y_7 = \frac{(7-2)(7-5)(7-8)}{(0-2)(0-5)(0-8)}\cdot 6 + \frac{(7-0)(7-5)(7-8)}{(2-0)(2-5)(2-8)}\cdot 10$$

$$+ \frac{(7-0)(7-2)(7-8)}{(5-0)(5-2)(5-8)}\cdot 12 + \frac{(7-0)(7-2)(7-5)}{(8-0)(8-2)(8-5)}\cdot 16$$

$$= \frac{5\cdot 2(-1)}{(-2)(-5)(-8)}\cdot 6 + \frac{7\cdot 2(-1)}{2\cdot(-3)(-6)}\cdot 10 + \frac{7\cdot 5(-1)}{5\cdot 3(-3)}\cdot 12 + \frac{7\cdot 5\cdot 2}{8\cdot 6\cdot 3}\cdot 16$$

$$y_7 = 0.125\times 6 - 0.389\times 10 + 0.778\times 12 + 0.486\times 16$$

$$= 0.75 - 3.89 + 9.336 + 7.776$$

$$= 13.972 \approx 14\text{ kgs}.$$

Thus, the weight of the 7 months baby is 14kgs.

1. Apply Lagrange's formula to find $f(6)$, given $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(7)=128$.

Soln: Here, $x=6$, $x_0=1$, $x_1=2$, $x_2=3$, $x_3=7$ and corresponding $y_0=2$, $y_1=4$, $y_2=8$ & $y_3=128$.

By Lagrange's interpolation formula

$$f(6) = \frac{(6-2)(6-3)(6-7)}{(1-2)(1-3)(1-7)} \times 2 + \frac{(6-1)(6-3)(6-7)}{(2-1)(2-3)(2-7)} \times 4 + \frac{(6-1)(6-2)(6-7)}{(3-1)(3-2)(3-7)} \times 8 + \frac{(6-1)(6-2)(6-3)}{(7-1)(7-2)(7-3)} \times 128$$

$$= \frac{4\cdot 3(-1)}{(-1)(-2)(-6)} \times 2 + \frac{5\cdot 3(-1)}{1(-1)(-5)} \times 4 + \frac{5\cdot 4(-1)}{2\cdot 1(-4)} \times 8 + \frac{5\cdot 4\cdot 3}{6\cdot 5\cdot 4} \times 128$$

$$= 1\times 2 - 3\times 4 + 2.5\times 8 + 0.5\times 128$$

$$= 2 - 12 + 20 + 64$$

$$f(6) = 74$$

2. Newton's Divided difference formula for unequal intervals.

Let $y=f(x)$ be the function which takes the values $y_0, y_1, y_2 \ldots$ corresponding to the arguments $x_0, x_1, x_2, \ldots x_n$ then the Newton's divided difference formula is given by (or Newton's general interpolation formula with divided difference).

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2) \ldots + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\Delta^n f(x_0).$$

1. By means of Newton's divided difference formula find the values of $f(8)$ and $f(15)$ from the table.

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Soln Divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 = 4$	$f(x_0) = 48$				
		$\frac{100-48}{5-4} = 52$			
$x_1 = 5$	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$	
$x_2 = 7$	294		$\frac{202-97}{10-5} = 21$		$\frac{1-1}{11-4} = 0$
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
$x_3 = 10$	900		$\frac{310-202}{11-7} = 27$		$\frac{1-1}{13-5} = 0$
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
$x_4 = 11$	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
$x_5 = 13$	2028				

Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots$$

$$\therefore f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1) + 0.$$

for $x = 8$

$$f(8) = 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) = \underline{448}$$

and $x = 15$ in above polynomial.

$$f(15) = 48 + (11)(52) + (11)(10)(15) + (11)(10)(8)(1) = \underline{3150}$$

2. Find the polynomial of lowest degree which assumes the values 10, 4, 40, 424 and 620 at $x = -2, 1, 3, 7$ and 8. Also find the value of the polynomial at $x = 6$.

Soln The Newton's divided difference table for the data is as follow:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 = -2$	$f(x_0) = 10$	$\frac{4-10}{1-(-2)} = -2$	$\frac{18-(-2)}{3-(-2)} = 4$	$\frac{13-4}{7-(-2)} = 1$	$\frac{1-1}{8-(-2)} = 0.$
$x_1 = 1$	4	$\frac{40-4}{3-1} = 18$	$\frac{96-18}{7-1} = 13$	$\frac{20-13}{8-1} = 1$	
$x_2 = 3$	40	$\frac{424-40}{7-3} = 96$	$\frac{196-96}{8-3} = 20$		
$x_3 = 7$	424	$\frac{620-424}{8-7} = 196$			
$x_4 = 8$	620				

The divided difference formula is

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$= 10 + (x+2)(-2) + (x+2)(x-1)(4) + (x+2)(x-1)(x-3)(1) + 0$$

$f(x) = x^3 + 2x^2 - 3x + 4$. is the polynomial

Also, for $x = 6$

$\therefore f(6) = 6^3 + 2\cdot 6^2 - 3\cdot 6 + 4 = \underline{\underline{274}}$

$R - (-1,1)$ $[0, \pi]$

Tan^{-1} R

cot^{-1} $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right.$

3. Find a polynomial of lowest possible degree by using Newton's Divided difference formula for the data:

x:	0	1	4	5
$f(x)$:	8	11	68	123.

Soln: The divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$x_0 = 0$	$f(x_0)=8$	$\frac{11-8}{1-0} = 3$	$\frac{19-3}{4-0} = 4$	$\frac{9-4}{5-0} = 1$
$x_1 = 1$	11	$\frac{68-11}{4-1} = 19$	$\frac{55-19}{5-1} = 9$	
$x_2 = 4$	68	$\frac{123-68}{5-4} = 55$		
$x_3 = 5$	123			

The Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$f(x) = 8 + (x-0)(3) + (x-0)(x-1)(4) + (x-0)(x-1)(x-4)(1)$$

$$= 8 + 3x + 4x(x-1) + (x^2-x)(x-4)$$

$$= 8 + 3x + \cancel{4x^2} - \cancel{4x} + x^3 - \cancel{4x^2} - x^2 + \cancel{4x}$$

$f(x) = x^3 - x^2 + 3x + 8$, is the required polynomial.

Numerical differentiation.

The method of evaluation of the derivative of a function at some particular value of the independent variable, when set of values of the fn is given, which is known as Numerical differentiation.

Using forward difference formula to the fn $y=f(x)$

$$\therefore \left(\frac{dy}{dx}\right)_{x=x_0} = f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right].$$

$$\text{and } \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right].$$

1. Find $f'(6)$ and $f''(6)$ from the following data.

x:	2	4	6	8	10
$y=f(x)$:	4	12	19	52	84

Sol: The forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	4	8	-1	27	-54
4	12	7	26	-27	
6	19	33	-1		
8	52	32			
10	84				

We have

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

at $x_0 = 6$, $h = 2$

$$f'(6) = \frac{1}{2}\left[\cancel{8}\,33 - \frac{1}{2}(-1) + \cancel{\frac{1}{3}(27) - \frac{1}{4}(-54)}\right]$$

$$= \frac{1}{2}\left[33 + 0.5 + \cancel{\cdots 5}\right]$$

$$f'(6) = 16.75 = \underline{16.75}$$

Also,

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$x_0 = 6$

$$f''(6) = \frac{1}{2^2}\left[-1 - \cancel{27 + \frac{11}{12}(-54)}\right] = \frac{1}{4}\left[\cancel{\cdots}\right] = \underline{-0.25}$$

$$f''(6) = \cancel{\cdots} = \underline{-0.25}$$

2. The given values of $\sin\theta$ for different values of θ

θ:	$0°$	$10°$	$20°$	$30°$	$40°$
$\sin\theta$:	0.0000	0.1736	0.3420	0.5000	0.6428

Find the value of $\cos(10°)$.

Soln. The difference table.

θ	$\sin\theta$	Δ	Δ^2	Δ^3	Δ^4
0	0.0000				
		0.1736			
$\theta =$ 10 →	0.1736		−0.0052		
		→ 0.1684 →		−0.0052	
20	0.3420		−0.0104		0.0004
		0.1580		→ −0.0048	
30	0.5000		−0.0152		
		0.1428			
40	0.6428				

$\theta = 10^\circ = \frac{\pi}{18}$ & $h = 10^\circ = \frac{\pi}{18}$.

$$\left.\frac{d(\sin\theta)}{d\theta}\right|_{\theta=10^\circ} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{18}{\pi}\left[0.1684 - \frac{1}{2}(0.0104) + \frac{1}{3}(-0.0048)\right]$$

$$\cos(\theta) \text{ at } \theta = 10^\circ = \frac{18}{\pi}(0.1720) = 0.9850$$

3. Given.

x :	0.1	0.2	0.3	0.4
$f(x)$:	1.10517	1.22140	1.34986	1.49182

Find $f'(x)$ at $x = 0.4$.

Soln The difference table

x	$f(x)$	∇	∇^2	∇^3
0.1	1.10517			
		0.11623		
			0.01223	
0.2	1.22140			0.00127
		0.12846		
			0.01350	
0.3	1.34986			
		0.14196		
$x =$ 0.4	1.49182			

Now, by Backward difference formula (Since $x = 0.4$ at end)

$$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \cdots\right]$$

at $x_0 = 0.4$, $h = 0.1$

$$f'(0.4) = \frac{1}{0.1}\left[0.14196 + \frac{1}{2}(0.01350) + \frac{1}{3}(0.00127)\right]$$

$R - (-1, 1)$ $[0, \pi]$

Tan^{-1} R $(-\pi$

$f'(0.4) = 10[0.14196 + 0.00675 + 0.00042]$

$f'(0.4) = \underline{1.49133}$

4. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 54$ from the following table.

x:	50	51	52	53	54
y:	3.6840	3.7084	3.7325	3.7563	3.7798.

Soln: ~~Find~~ the backward difference table

x	$f(x)$	∇	∇^2	∇^3
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		
$x=54$	3.7798			

We have, $h = 1$, $x = 54$

$$\left(\frac{dy}{dx}\right)_{at\, x=54} = \frac{1}{h}\left[\nabla y + \frac{1}{2}\nabla^2 y + \frac{1}{3}\nabla^3 y + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=54} = \frac{1}{1}\left[0.0235 + \frac{1}{2}(-0.0003)\right] = \underline{\underline{0.0232}}$$

also

$$\left(\frac{d^2y}{dx^2}\right)_{x=54} = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \cdots\right] = \frac{1}{1^2}\left[-0.0003\right] = -0.0003$$

$= \underline{-0.0003}$

Numerical integration

Numerical integration is the process of evaluating a definite integral from a set of values of the integrand for then the numerical integration is known as 'Quadrature'

The general quadrature formula for the definite integral

$$\int_a^b y(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n^2}{3} - \frac{n}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^3}{4} - n^2 + n\right)\frac{\Delta^3 y_0}{3!} + \cdots\right] \quad \text{①}$$

Consider other quadrature rules by taking $n = 1, 2, 3, \ldots$

1. Trapezoidal rule:

Taking $n=1$, in equation ①

$$\therefore \int_{x_0=a}^{b=x_0+nh} y(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$ which is

known as 'Trapezoidal rule' for numerical integration.

2. Simpson's one-third rule :-

Taking $n=2$ in eqn ①.

$$\therefore \int_{a=x_0}^{b=x_0+nh} y(x)dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\cdots+y_{n-2}) + 4(y_1+y_3+y_5+\cdots+y_{n-1})\right]$$

This formula is known as Simpson's $\frac{1}{3}$rd rule and this rule gives the interval must be divided into even number of equal sub-intervals.

3. Simpson's one-eighth rule (or) three-eighth rule.

Taking $n=3$ in the eqn ①.

$$\therefore \int_{a=x_0}^{b=x_0+nh} y(x)dx = \frac{3}{8}h\left[(y_0+y_n) + 2(y_3+y_6+y_9+\cdots+y_{n-3}) + 3(y_1+y_2+y_4+y_5+y_7+\cdots+y_{n-4}+y_{n-2}+y_{n-1})\right]$$

is called Simpson's three-eighth rule.

$\sec^{-1}$	$R-(-1,1)$	$[0,\pi]-\{\frac{\pi}{2}\}$	
$\tan^{-1}$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	
$\cot^{-1}$	R		

4. Weddle's rule

Taking $n=6$ in eqn ① then

$$\int_{a=x_0}^{b=x_0+nh} y(x)\,dx = \frac{3}{10}h\Big[(y_0+y_n)+5(y_1+y_5+y_7+\cdots+y_{n-5}+y_{n-1})$$
$$+(y_2+y_4+y_8+\cdots+y_{n-4}+y_{n-2})+$$
$$2(y_6+y_{12}+\cdots+y_{n-6})+$$
$$6(y_3+y_9+\cdots+y_{n-3})\Big]$$ is Known as

weddle's rule. In this rule interval must be divided into multiple of 6 or 12.

SC

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule

(iii) Simpson's $\frac{3}{8}$ rule (iv) Weddle's Rule.

Soln Dividing the interval (0,6) into six parts each of width $h=1$

Let $f(x)=y=\frac{1}{1+x^2}$, then

x:	0	1	2	3	4	5	6
y:	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= 0.5[1.027+1.7946]$$

$$\int_0^6 \frac{1}{1+x^2}dx = 1.4108$$

(ii) Simpson's $\frac{1}{3}$ rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left[(1+0.027) + 2(0.2+0.0588) + 4(0.5+0.1+0.0385)\right]$$

$$= \frac{1}{3}\left[1.027 + 0.5176 + 2.554\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3662$$

(iii) Simpson's $\frac{3}{8}$th rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3}{8}h\left[(y_0+y_6) + 2y_3 + 3(y_1+y_2+y_4+y_5)\right] \cancel{+2(y_2+y_4)+4(y_1+y_3+y_5)}$$

$$= \frac{3}{8}\left[(1+0.027) + 2(0.1) + 3(0.5+0.2+0.0588+0.0385)\right]$$

$$= \frac{3}{8}\left[1.027 + 0.2 + 2.3919\right]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3571.$$

(iv) Weddle rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[(y_0+y_6) + 5(y_1+y_5) + (y_2+y_4) + 6y_3\right]$$

$$= \frac{3}{10}\left[(1+0.027) + 5(0.5+0.0385) + (0.2+0.0588) + 6(0\right]$$

$$= \frac{3}{10}\left[1.027 + 2.6925 + 0.2588 + 0.6\right]$$

$$= \frac{3}{10}\left[3.9783 + 0.6\right] = 1.3735$$

$$\int_0^6 \frac{dx}{1+x^2} = \cancel{1.1935}\ 1.3735$$

into

Sec^{-1} R $-(-1,1)$

Tan^{-1}

2. Evaluate $\int_0^6 \frac{1}{1+x}dx$ by using

(i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule

(iii) Simpson's $\frac{3}{8}$th rule (iv) weddle's rule.

Soln: Let $y = \frac{1}{1+x}$, taking $h=1$

x:	0	1	2	3	4	5	6
y:	1	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) Trapezoidal rule.

$$\int_0^6 y_x\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$\therefore \int_0^6 \frac{dx}{1+x} = \frac{1}{2}\left[(1+0.1429)+2(0.5+0.3333+0.25+0.2+0.1667)\right]$$

$$= 2.02145$$

(ii) Simpson's $\frac{1}{3}$rd rule.

$$\therefore \int_0^6 \frac{dx}{1+x} = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left[(1+0.1429)+2(0.3333+0.2)+4(0.5+0.25+0.1667)\right]$$

$$= \frac{1}{3}\left[1.1429+1.0666+3.6668\right]$$

$$\int_0^6 \frac{dx}{1+x} = 1.9588$$

(iii) Simpson's $\frac{3}{8}$th rule.

$$\int_0^6 \frac{dx}{1+x} = \frac{3h}{8}\left[(y_0+y_6)+2y_3+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(1+0.1429)+2(0.25)+3(0.5+0.3333+0.2+0.1667)\right]$$

$$= \frac{3}{8}\left[1.1429+0.5+3.6\right]$$

$$\int_0^6 \frac{dx}{1+x} = \underline{\underline{1.9661}}$$

(iv) Weddle's Rule.

$$\int_0^6 \frac{dx}{1+x} = \frac{3h}{10}\left[(y_0+y_6)+5(y_1+y_5)+(y_2+y_4)+6y_3\right]$$

$$= \frac{3}{10}\left[(1+0.1429)+5(0.5+0.1667)+(0.3333+0.2)+6(0.25)\right]$$

$$= \frac{3}{10}\left[1.1429+3.3335+0.5333+1.5\right]$$

$$\int_0^6 \frac{dx}{1+x} = \underline{\underline{1.9529}}$$

A/M-19

3. Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_1^5 f(x)dx$, given that

3M

x:	1	2	3	4	5
$f(x)$:	13	50	70	80	100
	y_0	y_1	y_2	y_3	y_4

Sol: Simpson's $\frac{1}{3}$rd rule, $h=1$

$$\therefore \int_1^5 f(x)dx = \frac{h}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$= \frac{1}{3}\left[(13+100)+2(70)+4(50+80)\right]$

$= \frac{1}{3}\left[113+140+520\right] = 257.6667 \approx \underline{\underline{258}}$

A/M-19

4. Evaluate $\int_{-\frac{\pi}{2}}^{\pi/2} \cos x \, dx$ taking '6' sub-intervals by Weddle's Rule.

5M

Soln: Let $y = f(x) = \cos x$. Now, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-(-\frac{\pi}{2})}{6} = \frac{\pi}{6} = 30^\circ$

x:	-90°	-60°	-30°	0°	30°	60°	90°
y:	0	0.5	0.866	1	0.866	0.5	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$\therefore \int_{-\frac{\pi}{2}}^{\pi/2} \cos x \, dx = \frac{3}{10}h\left[(y_0+y_6)+5(y_1+y_5)+(y_2+y_4)+6y_3\right]$

$= \frac{\cancel{3}}{10} \cdot \frac{\pi}{\cancel{6}_2}\left[(0+0)+5(0.5+0.5)+(0.866+0.866)+6(1)\right]$

$= \frac{1}{20} \cdot \frac{22}{7}\left[5+1.732+6\right]$

$= \frac{11}{70}(12.732)$

$\int_{-\frac{\pi}{2}}^{\pi/2} \cos x \, dx = 2.00074 \approx \underline{\underline{2}}$

SWAMI VIVEKAN

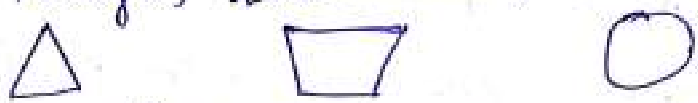
complex integration. (1)

Introduction:-

Connected set:- A set 'S' of points in Argand plane is said to be "connected set" if any two of its points can be joined by a continuous curve, all of whose points belong to 'S'.

simple curve:- A curve 'C' is said to be a simple curve if it does not intersect itself any where in the given region.

Ex:- Triangle, quadrilateral, circle.



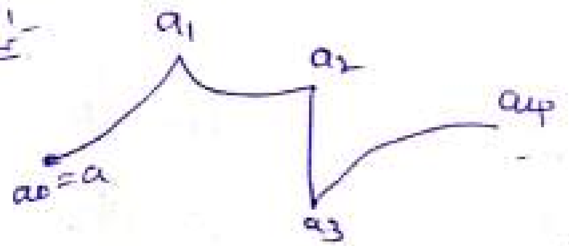
Example for Not simple curve:-

Simply connected region:- A region in which every closed curve can be shrunk to a point without passing out of the region is called simply connected region

Smooth curve:- A curve 'γ' is said to be smooth curve in a region 'R' if it is differentiable in each & every point of 'R'. (i.e $\gamma'(t) \neq 0$ in R if $\gamma(t) = x(t) + iy(t)$).

Piecewise smooth curve:- A curve 'γ' is said to be piecewise smooth curve on $[a,b]$ if $\exists$ a partition $a_0 = a_0 < a_1 < a_2 < \ldots < a_n = b$ $\ni$: γ is smooth on each subinterval $[a_i, a_{i+1}]$, $i = 1, 2, 3, \ldots, n$.

Ex:-



Here, whole curve is need not be smooth curve

(a_0, a_1), (a_1, a_2), (a_2, a_3) --- in each subinterval the curve is smooth. (i.e piecewise smooth curve).

Contour:-

A curve consisting of a finite no. of smooth curves joined end to end is called a contour.

Ex:- Triangle, rectangle, circles etc.

(or)

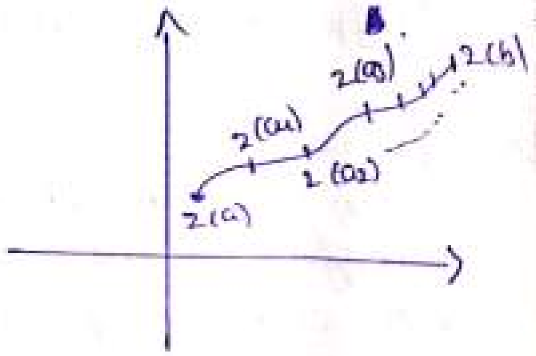
A piecewise smooth curve is called contour.

complex Line integral:-

let $f(z)$ is a fun defined & continuous on 'D' & $z(t)$ be the contour lying completely in 'D'.

i.e. $C: z(t) = x(t) + i\,y(t),\ t \in [a,b]$.

let $a_0 = a < a_1 < a_2 < \cdots < a_n = b$ be the partition of $z(t)$ in $[a,b]$.



Here, we choose a point 'b_k' on each arc joining z_{k-1} to z_k.

then the we form the sum,

$$s_n = \sum_{r=1}^{n} f(b_r)(z_r - z_{r-1}).$$

then the sum s_n tends to a fixed limit which does not depend upon the mode of subdivision and denote this limit by,

$\int_a^b f(z)\,dz$ (or) $\int_C f(z)\,dz$. which is called the complex line integral (or) line integral of $f(z)$

Examples for complex line integral :- (3)

Evaluate, $\int_c (\bar{z})^2 dz$, where 'c' is the circle (i) $|z|=r$.
(ii) $|z-2|=3$.

Sol. :- (i) Given $|z|=r$. is the ck with centre at $(0,0)$ and radius equal to 'r'.

then we have, $|z|=r \Rightarrow z = re^{i\theta}$, $0\le\theta\le 2\pi$.
$dz = re^{i\theta}(+i)\,d\theta$.

consider,

$$\int_c (\bar{z})^2 dz = \int_0^{2\pi} (re^{-i\theta})^2 (r)(+i) e^{+i\theta} d\theta.$$

$$= r^2(+i)\int_0^{2\pi} e^{-2i\theta}\cdot e^{i\theta} d\theta.$$

$$= ir^2 \int_0^{2\pi} e^{-2i\theta+i\theta} d\theta.$$

$$= ir^2 \int_0^{2\pi} e^{-i\theta} d\theta$$

$$= ir^2 \left.\frac{e^{-i\theta}}{-i}\right|_0^{2\pi}. \Rightarrow \frac{ir^2}{-i}(e^{-2\pi i}-1).$$

$$= -r^2(1-1) = 0.$$

(ii). $|z-2|=3$. The the ck with centre at $(2,0)$ and radius '3 units.

then, $|z-2|=3 \Rightarrow z-2 = 3e^{i\theta} \Rightarrow z = 2+3e^{i\theta}$.
$dz = 3ie^{i\theta}d\theta$.
$\bar{z} = 2+3e^{-i\theta}$.

$$\int_c (\bar{z})^2 dz = \int_0^{2\pi} (2+3e^{-i\theta})^2\, 3ie^{i\theta} d\theta.$$

$$= 3i\int_0^{2\pi} (2+3e^{-i\theta})^2 e^{i\theta} d\theta.$$

$$= 3i\int_0^{2\pi} (4+9e^{-2i\theta}+12e^{-i\theta})e^{i\theta} d\theta.$$

$$= 3i\int_0^{2\pi} (4e^{i\theta}+9e^{-i\theta}+12)\,d\theta.$$

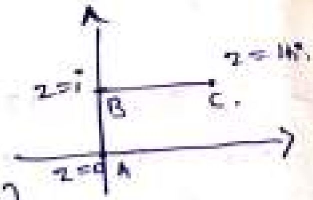
$$= 3i\left[4\frac{e^{i\theta}}{i} + 9\frac{e^{-i\theta}}{-i} + 12\theta\right]_0^{2\pi}$$

$$= 3i\left[\frac{4}{i}\cancel{e^{2\pi i}} + \frac{9}{-i}\cancel{e^{-2\pi i}} + 24\pi - \cancel{\frac{4}{i}} + \cancel{\frac{9}{i}} - 0\right]$$

$$= 72\pi i \quad //.$$

Q. Evaluate, $\int_C (y-x-3ix^2)\,dz$, where 'C' consists of two line segment from $z=0$ to $z=i$ and other from $z=i$ to $z=1+i$.

Sol. Here,



$$\int_C (y-x-3ix^2)\,dz = \int_{AB} + \int_{BC} \quad [\because c = AB+BC].$$

(4)

$$\int_{AB} (y-x-3ix^2)\,dz = \int_{AB} (y-x-3ix^2)(dx+idy).$$

along AB, $x=0 \Rightarrow dx=0$.

$$= \int_{AB} (y)\,i\,dy$$

$$= i\int_0^1 y\,dy \Rightarrow i\,\frac{y^2}{2}\Big|_0^1$$

$$\Rightarrow i/2.$$

$$\int_{BC} (y-x-3ix^2)\,dz = \int_{BC} (y-x-3ix^2)(dx+idy).$$

along BC, $y=1$, $dy=0$.

$$= \int_{BC} (1-x-3ix^2)(dx).$$

$$= \int_0^1 (1-x-3x^2 i)\,dx$$

$$= x - \frac{x^2}{2} - \frac{3x^3}{3} i \Big|_0^1$$

$$= 1 - \frac{1}{2} - 1i$$

$$\int_{BC} f(z)\,dz = +1/2 - i.$$

$$\therefore \int_C (y-x-3x^2 i)\,dz = \int_{AB} + \int_{BC}$$

$$= i/2 + 1/2 - i \quad = \frac{1}{2} - i/2.$$

5) Evaluate, $\int_0^{1+i} (x^2 - iy)\,dz$ along the path (i) $y = x^2$ (ii) $y = x$. ⑤.

Sol: (i) along $y = x^2$

$dy = 2x\,dx$. $0 \le x \le 1$

then, $\int_0^{1+i} (x^2 - iy)(dx + i\,dy) = \int_0^1 (x^2 - ix^2)(dx + i\,2x\,dx)$.

$= \int_0^1 (x^2 - ix^2)(1 + i2x)\,dx$.

$= (1-i)\int_0^1 x^2(1+i2x)\,dx$

$= (1-i)\int_0^1 (x^2 + i2x^3)\,dx$.

$= (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$

$= (1-i)\left[\frac{1}{3} + i/2\right]$.

$\int_0^{1+i} (x^2 - iy)\,dz = \frac{5+i}{6}$.

(ii). along $y = x$. $dy = dx$. then, $\int_0^{1+i} (x^2 - iy)(dx + i\,dy)$

$= \int_0^{1+i} (x^2 - ix)(dx + i\,dx) = \int_0^{1+i} (x^2 - ix)(1+i)\,dx$

$= (1+i) \int_0^1 (x^2 - ix)\,dx$

$= (1+i) \cdot \left(\frac{x^3}{3} - i\frac{x^2}{2} \right)_0^1$

$= (1+i)\left(\frac{1}{3} - i/2\right)$

$\int_0^{1+i} (x^2 - iy)\,dz = \frac{1}{6}(5 - i)$ //.

Evaluate, $\int_C \frac{z+2}{z}\,dz$, where "c" is (6).

(i) The semi circle $z = 2e^{i\theta}$. $-\pi \le \theta \le 0$

(ii) The circle $z = 2e^{i\theta}$. $-\pi \le \theta \le \pi$.

Sol (i) $\int_C \frac{z+2}{z}\,dz = \int_{-\pi}^{0} \frac{2e^{i\theta}+2}{\cancel{2}\cancel{e^{i\theta}}} \cancel{2} i \cancel{e^{i\theta}}\,d\theta.$ $\quad z = 2e^{i\theta}$, $dz = 2ie^{i\theta}\,d\theta$.

$$= 2i \int_{-\pi}^{0} (e^{i\theta}+1)\,d\theta.$$

$$= 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_{-\pi}^{0}$$

$$= 2i\left(\frac{1}{i} - \frac{e^{-i\pi}}{i} + \pi\right).$$

$$= 2i(-i+1+\pi)$$

$$\int_C \frac{z+2}{z}\,dz = +2+2i+2\pi i \text{ (or) } 2+(1+\pi)2i.$$

(ii) $\int_C \frac{z+2}{z}\,dz = \int_{-\pi}^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} \cancel{2} i \cancel{e^{i\theta}}\,d\theta.$

$$= 2i \int_{-\pi}^{\pi} (e^{i\theta}+1)\,d\theta$$

$$= 2i\left(\frac{e^{i\theta}}{i}+\theta\right)_{-\pi}^{\pi}$$

$$= \frac{2i}{i}\left(e^{i\theta}+\theta i\right)_{-\pi}^{\pi}$$

$$= 2\left(e^{i\pi}+\pi i - e^{-i\pi}+\pi i\right)$$

$$= 2(1+\pi i - 1 + \pi i)$$

$$= 4\pi i \;//.$$

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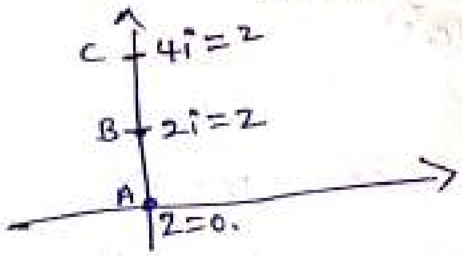
Evaluate, $\int_C \bar{z}\, dz$, where "C" is made up of the line from $z=0$ to $z=2i$ and the line from $z=2i$ to $z=4i$. (7).

(or) $(0,0)$ (or) $(0,2)$ (or) $(0,2)$ (or) $(0,4)$. y

Sol: Given. $\int_C \bar{z}\, dz$, C:



consider

$$\int_C \bar{z}\, dz = \int_{AB} + \int_{BC}.$$

$\int_{AB} (x - iy)(dx + i dy)$, along AB, $x = 0$. $dx = 0$.

then. $\int_{AB} (x-iy)(dx+idy) = \int_0^2 (-iy)(idy)$

$= \int_0^2 y\, dy \Rightarrow \frac{1}{2} y^2 \Big|_0^2$

$\Rightarrow 2.$

$\int_{BC}$, along BC, $x=0$, $dx=0$.

then $\int_{BC} (-iy)(idy) = \int_2^4 y\, dy \Rightarrow \frac{y^2}{2}\Big|_2^4$

$= \frac{1}{2}(16-4)$

$= 6.$

$$\therefore \int_C \bar{z}\, dz = \int_{AB} + \int_{BC}$$

$= 2+6$

$= 8$ //

Evaluate, $\int_{(0,0)}^{(3,1)} z^2\, dz$ (or) $\int_{z=0}^{3+i} z^2\, dz$ along the line $y = x$.

Sol: Consider $\int_0^{3+i} z^2\, dz = \frac{z^3}{3}\Big|_0^{3+i}.$

$= \frac{(3+i)^3}{3}$ //

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Evaluate,

$$\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy \quad \text{using the pharametric curves, } x = 2t,\ y = t^2+3.$$ (8).

Sol Given. $x = 2t,\quad y = t^2+3$

$dx = 2dt,\quad dy = 2t\,dt.$

$t = x/2,$
$x=0.\ t=0.$
$x=2\ \ t=1.$
$t = \pm\sqrt{y-3}.$
$y=3,\ t=0.$
$y=4,\ t=1.$
$\therefore 0 \le t \le 1$

then, $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$

$= \int_0^1 (2t^2+6+4t^2)\,2dt + (6t-t^2-3)\,2t\,dt.$

$= 2\int_0^1 (2t^2+6+4t^2+6t^2-t^3-3t)\,dt.$

$= 2\int_0^1 (-t^3+12t^2-3t+6)\,dt.$

$= 2\left(-\frac{t^4}{4}+4t^3-3\frac{t^2}{2}+6t\right)_0^1$

$= 2\left(-\frac{1}{4}+4-\frac{3}{2}+6\right). \Rightarrow 2\left(\frac{-1+16-6+24}{4}\right) = \frac{33}{2}$ //.

Important problems on line Integral:-

(i) Evaluate, $\int_C (x^2-iy^2)\,dz$ along $y=2x^2$. from $(1,2)$ to $(2,8)$.

(ii) Evaluate. $\int_C (\bar{z})^2\,dz$ around the circle (i) $|z|=1$, (ii) $|z-1|=1$.

(iii) Evaluate. $\int_C \bar{z}\,dz$, when 'c' is given by two line segment joining $z=0$ to $z=2i$ and $z=2i$ to $z=4+2i$

(iv). Show that,
(i) $\int_C \frac{dz}{z-a} = 2\pi i$ (ii) $\int_C \frac{dz}{(z-a)^n} = 0,\ n=2,3,4,\ldots$ where 'c' is $|z-a|=r$.

Evaluate, $\int_{i}^{3+10i} (3x+y)\,dx + (2y-x)\,dy$ along the line joining the points $(0,1)$ to $(3,10)$. (9).

Sol. Here, $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (3,10)$.

The line joining these two points is given by,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x-0}{3} = \frac{y-1}{10-1}$$

$$\Rightarrow \frac{x}{3} = \frac{y-1}{9}$$

$\Rightarrow y = 3x+1$. then $dy = 3dx$.

then, $\int_0^3 (3x+3x+1)\,dx + (6x+2-x)(3dx) = \cancel{\int_0^3 (11x+3)\,dx}$

$\cancel{= 11\frac{x^2}{2} + 3x \Big|_0^3}$

$\cancel{= \frac{11 \cdot 9}{2} + 9}$

$\cancel{= \frac{99}{2} + 9 \Rightarrow \frac{99+18}{2}} \Rightarrow$

$$= \int_0^3 (21x+7)\,dx.$$

$$= 21\frac{x^2}{2} + 7x \Big|_0^3$$

$$= \frac{21}{2} \times 9 + 21$$

$$= 231/2 \text{ //.}$$

properties of line integral:-

(i) $\int_C k f(z)\,dz = k\int_C f(z)$, $k \in \mathbb{R}$ (const.)

(ii) $\int_C (f(z) \pm g(z))\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$.

(iii) $\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$.

(iv) C is contain $C_1, C_2, \dots C_n$ i.e. $C = C_1 + C_2 + C_3 + \dots + C_n$

then $\int_C f(z)\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz + \dots + \int_{C_n} f\,dz$.

Note:-

Green's theorem:-

If $P(x,y)$ & $Q(x,y)$ be two continuous fu^n having continuous first order derivative in the region 'D' consisting of points within & on the simple closed contour 'C', then

$$\int_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Cauchy's Integral theorem:-

statement:- If a fu^n $f(z)$ be analytic at all points within & on closed contour 'C', then

$$\int_C f(z)\,dz = 0.$$

proof:- let the domain enclosed by 'C' denoted by 'D'

Also, $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$.

consider $\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$

$$= \int_C (udx - vdy) + i\int_C vdx + udy \rightarrow ①.$$

By data $f(z)$ is analytic and hence $f'(z)$ exists and the four pde $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, continuous in 'D' and on 'C'. They the four pde $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, continuous in 'D' and on 'C'.

$\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continous in 'D' & on 'C'.

$\therefore$ By Green's theorem,

$$\int_C udx + (-v)dy = \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy.$$

Since $f(z)$ is analytic by Green's then $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\Rightarrow \int_C udx + (-v)dy = \iint_D \left(-\cancel{\frac{\partial v}{\partial x}} + \cancel{\frac{\partial v}{\partial x}}\right) dx\,dy$$

$$= 0. \rightarrow ②.$$

Again, $\int_C (v\,dx + u\,dy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$

$$= \iint_D \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dx\,dy \quad \left(\because \text{ By C-R eq. } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$$

$$\therefore \int_C (v\,dx + u\,dy) = 0 \longrightarrow (3)$$

Using (2) & (3) in (1).

$$\int_C f(z)\,dz = 0.$$

Cauchy's Integral formula:-

Statement:- If $f(z)$ be analytic within and on a closed contour C of a simply connected region & $z = z_0$ is an interior point of C, then $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$.

Proof:- By data $f(z)$ is analytic inside and on C.

Thus $\phi(z) = \frac{f(z)}{z-z_0}$ is also analytic inside and on C, except at $z = z_0$. ($\because$ at z_0 $\phi(z)$ is not defined).

Let us draw a circle C_1 with center at z_0 and radius r, lying entirely inside C.

Now any point on C_1 is given by

$z = z_0 + re^{i\theta}$ and therefore $dz = ire^{i\theta}\,d\theta$.

Also the fun $\phi(z)$ is analytic in the region enclosed between C & C_1. Hence by second consequence of cauchy's integral we have,

$$\int_C \phi(z)\,dz = \int_{C_1} \phi(z)\,dz.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0} dz = \int_{C_1} \frac{f(z)}{z-z_0} dz.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0} dz = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta.$$

$$\int_C \frac{f(z)}{z-z_0} dz = i\int_0^{2\pi} f(z_0+re^{i\theta}) d\theta. \longrightarrow (1)$$

Now as $r \to 0$, the ck c_1 tends to a point.

Thus as $r \to 0$, the eq (1) becomes

$$\int_C \frac{f(z)}{z-z_0} dz = i\int_0^{2\pi} f(z_0) d\theta.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0} dz = i f(z_0) \int_0^{2\pi} d\theta.$$

$$\Rightarrow \int_C \frac{f(z)}{(z-z_0)} dz = i f(z_0) \cdot 2\pi.$$

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz.$$ Hence the proof

Note:- The cauchy's integral formula gives the value of an analytic fu^n at any interior point of a simply connected region in terms of its value on the boundry of the region.

Cauchy's integral formula for derivative of a fu^n:-

Statement:- If $f(z)$ is analytic within & on a closed curve 'c' & $z=z_0$ is an interior point of 'c' then

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$ when $n \in I^+$.

Proof:- we have by Cauchy's integral formula.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz.$$

Now diff. above wrt z_0 by using Leibnitz rule.

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial z_0} \frac{f(z)}{(z - z_0)}\, dz$$

$$= \frac{1}{2\pi i} \oint_C f(z) \cdot \frac{-1}{(z - z_0)^2} (-1)\, dz.$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz.$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{1+1}}\, dz.$$

Again Applying Leibnitz rule,

$$f''(z_0) = \frac{1}{2\pi i} \oint_C f(z) \cdot \frac{-2}{(z - z_0)^3} (-1)\, dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{2+1}}\, dz.$$

continue like this

$$f'''(z_0) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{3+1}}\, dz.$$

⋮ After nth differentiation.

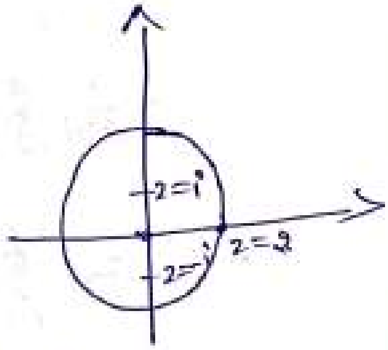
$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$ when $n \in I^+$.

Hence the proof.

Evaluate, $\int_C \frac{z e^z}{z^2+1} dz$, where "c" is the ⊙le $|z|=2$.

Sol :- Given $|z|=2$ is the ⊙le with center at $z=0$ and radius equal to "2".

Now, $z^2+1=0 \Rightarrow z=\pm i$. lies within the circle $|z|=2$.



By partial fran.

$$\frac{1}{z^2+1} = \frac{i/2}{z+i} - \frac{i/2}{z-i}$$

then $\int_C \frac{z e^z}{z^2+1} dz = \frac{i}{2} \int \frac{z e^z dz}{z+i} - \frac{i}{2} \int \frac{z e^z}{z-i} dz$.

Since $f(z) = z e^z$, $f'(z) = z e^z + e^z$ is also continous. since "i" & "-i" are inside.

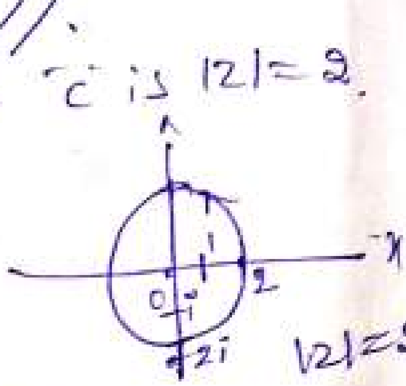
$$\therefore \int_C \frac{z e^z}{z^2+1} dz = \frac{i}{2} \cdot 2\pi i f(-i) - \frac{i}{2} \cdot 2\pi i f(i)$$

$$= \frac{i}{2} \cdot 2\pi i (-i) e^{-i} - \frac{i}{2} 2\pi i \cdot i \cdot e^{i}$$

$$= 2\pi i \left(\frac{e^{i} + e^{-i}}{2} \right)$$

$$= 2\pi i \cos(1) \quad //.$$

Evaluate, $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$, when "c" is $|z|=2$.



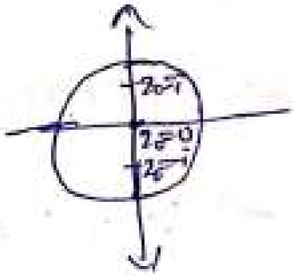
$|z|=2$.

Sol. $f(z) = \frac{z}{9-z^2}$.

then $\int \frac{z\,dz}{(9-z^2)(z+i)} dz = 2\pi i f(-i)$.

$$= 2\pi i \left[\frac{-i}{9-(-i)^2} \right] = \pi/5 //.$$

Evaluate, $\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz$, when 'c' is $|z| = 2.5$.



Sol Here. $f(z) = \cosh(\pi z)$.

$$\frac{1}{z(z^2+1)} = \frac{1}{z(z+i)(z-i)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}.$$

$$A = \frac{1}{(z-i)(z+i)} = 1 \text{ at } z = 0.$$

$$B = \frac{1}{z(z+i)} = -1/2 \text{ at } z = i.$$

$$C = \frac{1}{z(z-i)} = -1/2 \text{ at } z = -i.$$

Here, $z_0 = 0$, $z_0 = +i$, $z_0 = -i$ are points inside 'c'.

then $\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz = 2\pi i [A f(0) + B f(i) + C f(-i)]$

$$= 2\pi i \left(\cos(0) - \frac{1}{2} \cos(i^2\pi) + \frac{1}{2} \cos(i^2\pi) \right]$$

$$= 2\pi i [1 + 1/2 + 1/2] = 4\pi i //.$$

Cauchy's inequality:-

Statement:- If $f(z)$ is analytic within & on the circle 'c' with center at $z = a$ and radius 'r', then

$|f^n(a)| \le M \cdot \frac{n!}{r^n}$, for $n = 0, 1, 2, 3 \ldots$ where 'M' is a +ve. number such that $|f(z)| \le M \;\forall\; z \in C$.

proof:- By cauchy's integral formula, $f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$.

Take, $z_0 = a$.

then, $f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$.

Take modulus on both side.

then. $|f^n(a)| = \frac{|n!|}{|2\pi i|} \oint_C \frac{|f(z)|}{|(z-a)^{n+1}|} |dz|$.

$$\Rightarrow |f^n(a)| \le \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^{n+1}} |re^{i\theta} d\theta|$$

$$\Rightarrow |f^n(a)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M}{r^{n+1}} \cdot r \cdot d\theta$$

$$\Rightarrow |f^n(a)| \leq \frac{n!}{2\pi} M \int_0^{2\pi} \frac{1}{r^n} d\theta$$

$$\Rightarrow |f^n(a)| \leq \frac{n!}{2\pi} M \frac{1}{r^n} (\theta)_0^{2\pi}$$

$$\Rightarrow |f^n(a)| \leq \frac{n!}{\cancel{2\pi}} \frac{M}{r^n} \cancel{2\pi}.$$

$\therefore |f^n(a)| \leq M \frac{n!}{r^n}$ // is the required cauchy's inequality

Liouville's theorem :-

statement :- If $f(z)$ be analytic and bounded in the entire complex plane then $f(z)$ is constant.

proof:- By defn $f(z)$ is bounded $\forall z$ of the complex plane. Thus ~~three~~ exists a constant $M \ni: |f(z)| \leq M$ $\forall z$. In particular $|f(z)| \leq M$ $\forall z$ in and on the circle c with center at $z = a$ and radius r.

$\therefore$ By cauchy's inequality we have,

$$|f'(a)| \leq \frac{M}{r}.$$

Now, as $r \to \infty$, then $\frac{M}{r} \to 0$.

Thus as $r \to \infty$ $|f'(a)| = 0$.

ie $f'(a) = 0$.

But a is arbitrary. Thus $f'(z) = 0$ $\forall z$.

Hence $f(z)$ is constant.

Hence the result.

Fundamental theorem of algebra:-

Every polynomial equation

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n = 0.$

where $n \geq 1$ and $a_n \neq 0$, has atleast one root.

Proof:- suppose $f(z) = 0$ has no roots.

$\Rightarrow f(z) \neq 0 \ \forall z.$

$\Rightarrow g(z) = \frac{1}{f(z)}$ is analytic $\forall z$.

Also $g(z) = \frac{1}{f(z)} \longrightarrow 0$ as $z \longrightarrow \infty$.

$\Rightarrow g(z)$ is bounded $\forall z$.

Thus by Liouville's theorem $g(z)$ must be a constant.

$\Rightarrow f(z)$ is also a constant.

But this is contradiction to our assumption $f(z)$ is a polynomial.

$\therefore \exists z \ni : f(z) = 0.$

$\therefore f(z)$ has atleast one root.