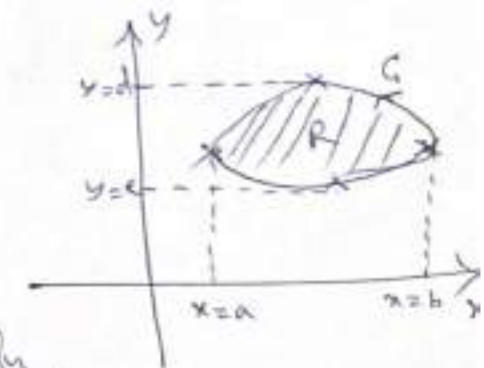
Integral theorems (i) II BSc Maths. (SVK)
(Paper - 4.1)

I. Green's theorem:

Statement: Let C be a simple closed curve bounded by a region R defined on the xy-plane. If $P(x,y)$ & $Q(x,y)$ be continuous and diff'ble functions on R.

Then $$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



proof: Let R be the region bounded by the closed curve C on the xy-plane. Let the lines $x=a$, $x=b$ & $y=c$, $y=d$ are parallel to x and y-axes respectively as shown in the figure.

Let C_1: APB and C_2: BQA be the curves denoted by the equations $y = f_1(x)$ and $y = f_2(x)$ respectively.

$$\therefore \iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} \left(\frac{\partial P}{\partial y}\right) dy\,dx = \int_{x=a}^{b} P(x,y)\Big|_{y=f_1(x)}^{f_2(x)} dx$$

$$= \int_{x=a}^{b} \big(P(x, f_2(x)) - P(x, f_1(x))\big)\,dx$$

$$= \int_{x=a}^{b} P(x, f_2(x))\,dx - \int_{x=a}^{b} P(x, f_1(x))\,dx$$

$$= -\int_{x=b}^{a} P(x, f_2(x))\,dx - \int_{x=a}^{b} P(x, f_1(x))\,dx$$

$$= -\left[\int_{x=b}^{a} P(x, f_2(x))\,dx + \int_{x=a}^{b} P(x, f_1(x))\,dx\right]$$

$$= -\left[\int_{C_2} P(x,y)\,dx + \int_{C_1} P(x,y)\,dx\right]$$

$$\iint_R \left(\frac{\partial P}{\partial y}\right) dx\,dy = -\oint_C P(x',y)\,dx .$$

$$\therefore \oint_C P(x,y)\,dx = -\iint_R \left(\frac{\partial P}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

Let $C_3 : QAP$ & $C_4 : PBQ$ be the Curves denoted by the equations $x = g_1(y)$ & $x = g_2(y)$ respectively

Then wehave

$$\iint_R \left(\frac{\partial Q}{\partial x}\right) dx\,dy = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} \left(\frac{\partial Q}{\partial x}\right) dy\,dx$$

$$= \int_{y=c}^{d} Q(x,y)\Big|_{x=g_1(y)}^{g_2(y)} dy$$

$$= \int_{y=c}^{d} \left(Q(g_2(y),y) - Q(g_1(y),y)\right) dy$$

$$= \int_{y=c}^{d} Q(g_2(y),y)\,dy - \int_{y=c}^{d} Q(g_1(y),y)\,dy$$

$$= \int_{y=c}^{d} Q(g_2(y),y)\,dy + \int_{y=d}^{c} Q(g_1(y),y)\,dy$$

$$= \int_{C_4} Q(g_2(y),y)\,dy + \int_{C_3} Q(g_1(y),y)\,dy$$

$$= \int_{C_4} Q(x,y)\,dy + \int_{C_3} Q(x,y)\,dy$$

$$\iint_R \left(\frac{\partial Q}{\partial x}\right) dx\,dy = \oint_C Q(x,y)\,dy$$

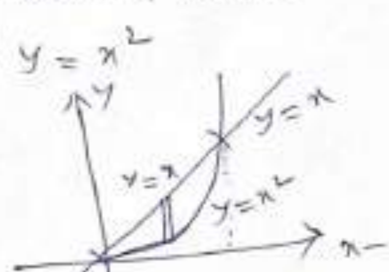
$$\therefore \int_C Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x}\right) dx\,dy \quad \text{--- (2)}$$

Eqns (1) + (2) we get

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hence the proof

Problems: ① Using Green's theorem evaluate $\oint_C [(xy+y^2)dx + x^2 dy]$ where C is the closed curve bounded by $y=x$ and $y=x^2$



Sol: By Green's theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore \oint_C [(xy+y^2)\,dx + x^2 dy] = \iint_R \left(\frac{\partial (x^2)}{\partial x} - \frac{\partial (xy+y^2)}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy$$

Let $C: y = x$ & $y = x^2$.

$y = x^2$, put $y = x$

$x = x^2 \therefore x^2 - x = 0$

$x(x-1) = 0$

$x = 0, x = 1$

$\therefore$ Curve C intersects at $(0,0)$ & $(1,1)$.

when $x = 0, y = 0$

$x = 1, y = 1$.

Limits are $x^2 \le y \le x$ & $0 \le x \le 1$

$$\oint_C [(xy+y^2)\,dx + x^2 dy] = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} (xy - y^2)\Big|_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left(x(x - x^2) - (x^2 - x^4)\right) dx$$

$$= \int_{x=0}^{1} (x^2 - x^3 - x^2 + x^4)\,dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)\Big|_{x=0}^{1}$$

$$\oint_C [(xy+y^2)\,dx + x^2 dy] = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}.$$

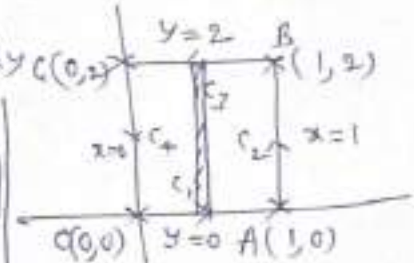
2) Using Green's theorem in the plane for evaluate:

$$\oint_C [(3x^2-8y^2)\,dx+(4y-6xy)\,dy]$$

Where C in the boundary of the rectangular area bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$

Sol: By Green's theorem.

$$\oint_C [P\,dx+Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\,dy$$



$$\oint_C (3x^2-8y^2)dx+(4y-6xy)\,dy = \iint_R \left[\frac{\partial(4y-6xy)}{\partial x}-\frac{\partial(3x^2-8y^2)}{\partial y}\right] dx\,dy$$

$$= \iint_R (-6y+16y)\,dy\,dx = \iint_R 10y\,dy\,dx.$$

Limits are $0 \le y \le 2$ & $0 \le x \le 1$.

$$\oint_C [(3x^2-8y^2)dx+(4y-6xy)\,dy] = \int_{x=0}^{1}\int_{y=0}^{2} 10y\,dy\,dx$$

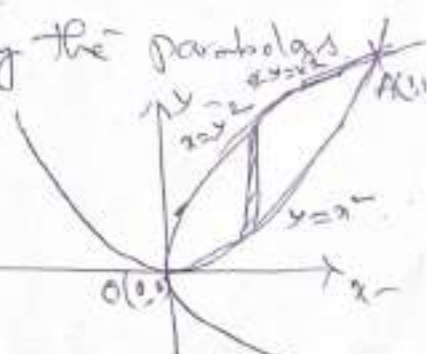
$$= \int_{x=0}^{1} (5y^2)\Big|_{y=0}^{2} dx$$

$$= \int_{x=0}^{1} 5(2^2-0)\,dx = 20\int_0^1 dx = 2(1-0)$$

$$\oint_C [(3x^2-8y^2)dx+(4y-6xy)dy] = 20.$$

3) Using Green's theorem evaluate $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$

Where C in the region bounded by the parabolas $y^2=x$ & $y=x^2$



Sol: $C: y=x^2$ & $x=y^2$

$\therefore x = x^4 \therefore (x^4-x)=0$

$\therefore x(x^3-1)=0.$

$\therefore x=0$ & $x^3-1=0.$ $\quad x=0$ & $(x-1)(x^2+x+1)=0$

$x=0$ & $x-1=0 \Rightarrow x=0$ & $x=1$

When $x=0$ then $y=1$ & $x=1$, then $y=1$

Curve C intersects at $(0,0)$ & $(1,1)$

$\therefore$ Limits are $0 \le x \le 1$ & $x^2 \le y \le \sqrt{x}$.

$\therefore$ By Green's theorem we have

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy] = \iint_R \left(\frac{\partial (4y-6xy)}{\partial x} - \frac{\partial (3x^2 - 8y^2)}{\partial y}\right) dx\,dy$$

$$= \iint_R (-6y + 16y)\,dx\,dy$$

$$= \iint_R 10y\,dy\,dx = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx$$

$$= \int_{x=0}^{1} 5y^2 \Big|_{y=x^2}^{\sqrt{x}} dx = \int_{x=0}^{1} 5(x - x^4)\,dx.$$

$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \left(\frac{5}{2}x^2 - 5\frac{x^5}{5}\right)\Big|_{x=0}^{1} = \frac{5}{2} - 1 = \frac{3}{2}$$

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4) Using Green's theorem evaluate

$\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where C is the rectangle

with vertices $(0,0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$ & $(0, \frac{\pi}{2})$.

Sol: By Green's theorem

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

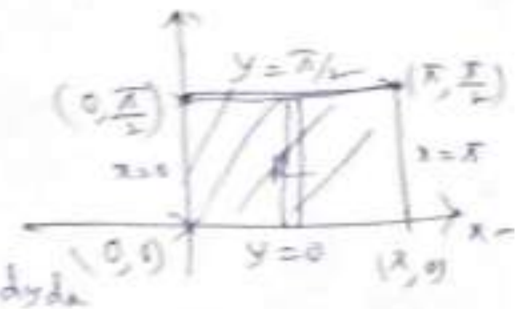
$$\oint_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy) = \iint_R \left(\frac{\partial (e^{-x}\cos y)}{\partial x} - \frac{\partial (e^{-x}\sin y)}{\partial y}\right) dx\,dy$$

$$= \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy) = \iint_R -2e^{-x}\cos y\,dx\,dy$$

Let C be the rectangle with vertices are (0,0), $(\pi,0)$, $(\pi,\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$

Limits are $0 \le x \le \pi$, $0 \le y \le \pi/2$



$$\therefore \oint_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy) = \iint_R -2e^{-x}\cos y\,dy\,dx$$

$$= -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} e^{-x}\cos y\,dy.dx$$

$$= -2\int_{x=0}^{\pi} e^{-x}(\sin y)\Big|_{y=0}^{\pi/2} dx \qquad \Big|\ \sin\pi/2 = 1,\ \sin 0 = 0$$

$$= -2\int_0^{\pi} e^{-x}(\sin\pi/2 - \sin 0)\,dx$$

$$= -2\int_0^{\pi} e^{-x}(1-0)\,dx = -2\int_0^{\pi} e^{-x}\,dx$$

$$= -2\left(\frac{e^{-x}}{-1}\right)\Big|_{x=0}^{\pi} = 2(e^{-\pi} - e^{-0}) = 2(e^{-\pi}-1)$$

5) Evaluate by Green's theorem in the plane

$\oint_C [(x^2 - \cosh y)dx + (y+\sin x)dy]$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$ & $(0,1)$.

Sol: By Green's theorem

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\oint_C [(x^2 - \cosh y)\,dx + (y+\sin x)\,dy] = \iint_R \left[\frac{\partial}{\partial x}(y+\sin x) - \frac{\partial}{\partial y}(x^2 - \cosh y)\right]dx\,dy$$

$$\int_C (x^2 - \cosh y)dx + (y+\sin x)dy = \iint_R (\cos x + \sinh y)\,dx\,dy.$$

Let C be the region bounded by the rectangle with vertices are $(0,0)$, $(\pi,0)$, $(\pi,1)$ & $(0,1)$

Limits are $0 \le y \le 1$ and $0 \le x \le \pi$.

$$\oint_C [(x^2 - \cosh y)\,dx + (y + \sin x)\,dy] = \iint_R \left(\frac{\partial(y+\sin x)}{\partial x} - \frac{\partial(x^2 - \cosh y)}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{1} (y\cos x + \cosh y)\Big|_{y=0}^{1} dx$$

$$= \int_{x=0}^{\pi} (\cos x + \cosh 1 - 1)\,dx \quad (\because \cosh 0 = 1)$$

$$= (\sin x + x(\cosh 1 - 1))\Big|_{x=0}^{\pi}$$

$$= (\sin\pi - \sin 0) + \pi(\cosh 1 - 1)$$

$$= \pi(\cosh 1 - 1) \quad (\because \sin\pi = \sin 0 = 0)$$

6) Using Green's theorem evaluate;

$$\oint_C [(y - \sin x)\,dx + \cos x\,dy]$$ where C is the triangle whose vertices are $(0,0), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, 1)$.

Sol: Let Region R bounded by the vertices $O(0,0)$, $A(\frac{\pi}{2}, 0)$ & $B(\frac{\pi}{2}, 1)$

$\therefore$ Limits are

$0 \le y \le \frac{2}{\pi}x$ & $0 \le x \le \pi/2$

Eqn of the line OB is, $y - 0 = \frac{(1-0)}{(\pi/2 - 0)}(x-0)$

$y = \frac{2}{\pi}x$

By Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C (y - \sin x)\,dx + \cos x\,dy = \iint_R \left(\frac{\partial(\cos x)}{\partial x} - \frac{\partial}{\partial y}(y - \sin x)\right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\iint_R (\sin x + 1)\,dx\,dy$$

$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2}{\pi}x}(\sin x+1)\,dy\,dx$$

$$= -\int_{x=0}^{\pi/2}(\sin x+1)(y)\Big|_{y=0}^{\frac{2}{\pi}x}dx$$

$$= -\int_{x=0}^{\pi/2}(\sin x+1)\frac{2}{\pi}x\,dx$$

$$= -\frac{2}{\pi}\int_{x=0}^{\pi/2}(x\sin x+x)\,dx$$

$$= -\frac{2}{\pi}\left[\int_{x=0}^{\pi/2}\underset{u}{x}\,\underset{v}{\sin x}\,dx+\int_{x=0}^{\pi/2}x\,dx\right]$$

$\int uv = u\int v - \int du\int v$. applying integration by parts.

$$= -\frac{2}{\pi}\left[(-x\cos x)\Big|_{x=0}^{\pi/2}+(\sin x)\Big|_{x=0}^{\pi/2}+\left(\frac{x^2}{2}\right)\Big|_{x=0}^{\pi/2}\right]$$

$$= -2/\pi\left[-(0-0)+1+\frac{\pi^2}{8}\right] = -\left[\frac{2}{\pi}+\frac{\pi}{4}\right]$$

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7) Using Green's theorem evaluate

$$\oint_C\left[(3x^2-8y^2)dx+2y(2-3x)dy\right]$$ where C is the curve enclosing the region bounded by the curve $y=x^2$ & the line $y=x$.

8) Using Green's theorem evaluate

$$\oint_C (x^2-xy^3)dx+(y^2-2xy)dy$$

where C is the square with vertices $(0,0),(2,0),(2,2)\,\&\,(0,2)$

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Gauss divergence theorem

Statement: If $\vec{F}(x,y,z)$ be a Continuously differentiable vector point function and V is the volume bounded by a closed surface S, then

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\cdot dv$$

where $\hat{n}$ is the outward drawn unit normal vector

Problems: ① Using Gauss divergence theorem evaluate for $\vec{F} = 2xyi + yz^2j + xzk$ and S is the total surface of the rectangular parallelopiped bounded by the planes $x=0, y=0, z=0, x=1, y=2, z=3$

Sol: By Gauss divergence theorem we have

$$\iint_R \vec{F}\cdot\hat{n}\cdot ds = \iiint_V \operatorname{div}\vec{F}\cdot dv$$

Let $\vec{F} = 2xyi + yz^2j + xzk$.

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(i\cdot\frac{\partial}{\partial x} + j\cdot\frac{\partial}{\partial y} + k\cdot\frac{\partial}{\partial z}\right)(2xyi + yz^2j + xzk)$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz) = (2y + z^2 + x)$$

$dv = dx\,dy\,dz$, $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$.

$$\iint_R \vec{F}\cdot\hat{n}\, ds = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3}(2y + z^2 + x)\, dz\, dy\, dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left(2yz + \frac{z^3}{3} + xz\right)\Big|_{z=0}^{3} dy\, dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}(6y + 9 + 3x)\, dy\, dx$$

$$= \int_{x=0}^{1}(3y^2 + 9y + 3xy)\Big|_{y=0}^{2} dx$$

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$$= \int_{x=0}^{1} (3(4) + 9(2) + 3x(2))\, dx$$

$$= \int_{x=0}^{1} (12+18+6x)\, dx$$

$$= \int_{x=0}^{1} (30+6x)\, dx = (30x+3x^2)\Big|_{x=0}^{1} = 30+3.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = 33$$

2) Using Gauss divergence theorem for $\vec{F} = y^2z^2 i + z^2x^2 j + z^2y^2 k$ and S is the total Surface area of the hemisphere $z \geq 0$, $x^2+y^2+z^2=1$.

Sol: By Gauss divergence theorem

$$\iint_R \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$$

$\vec{F} = y^2z^2 i + z^2x^2 j + z^2y^2 k$.

Let $\operatorname{div}\vec{F} = \left(i\cdot\frac{\partial}{\partial x} + j\cdot\frac{\partial}{\partial y} + k\cdot\frac{\partial}{\partial z}\right)(y^2z^2 i + z^2x^2 j + z^2y^2 k)$

$$= \frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)$$

$$\operatorname{div} F = 0+0+2zy^2 = 2zy^2. \quad dv = dx\, dy\, dz$$

Let $z \geq 0$, $x^2+y^2+z^2=1$

Let $z=0, y=0, x\neq 0, x^2=1 \Rightarrow x = \pm 1$.

Let $z=0, y\neq 0, x\neq 0$, $\quad x^2+y^2=1 \Rightarrow y = \pm\sqrt{1-x^2}$

Let $z\neq 0, y\neq 0, x\neq 0$, $\quad x^2+y^2+z^2=1$.

$$z = \sqrt{1-x^2-y^2}$$

$\therefore$

$$-1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad 0 \leq z \leq \sqrt{1-x^2-y^2}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} 2zy^2\, dz\, dy\, dx$$

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$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2 z^2) \Big|_{z=0}^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2(1-x^2-y^2)\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2(1-x^2)-y^4)\, dy\, dx$$

But $\displaystyle\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y^2(1-x^2)-y^4)\, dy = 2\int_0^{\sqrt{1-x^2}} (y^2(1-x^2)-y^4)\, dy$.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = 2\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} (y^2(1-x^2)-y^4)\, dy\, dx$$

$$= 2\int_{x=-1}^{1} \left(\frac{y^3}{3}(1-x^2) - \frac{y^5}{5}\right)\Big|_{y=0}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{x=-1}^{1} \left(\frac{(1-x^2)^{3/2}(1-x^2)}{3} - \frac{(1-x^2)^{5/2}}{5}\right) dx$$

$$= 2\int_{x=-1}^{1} \left(\frac{(1-x^2)^{5/2}}{3} - \frac{(1-x^2)^{5/2}}{5}\right) dx$$

$$= 2\int_{x=-1}^{1} (1-x^2)^{5/2}\left(\frac{1}{3}-\frac{1}{5}\right) dx$$

$$= 2\int_{x=-1}^{1} (1-x^2)^{5/2}\left(\frac{2}{15}\right) dx$$

$$= \frac{4}{15}\int_{x=-1}^{1} (1-x^2)^{5/2}\, dx = \frac{4}{15}\, 2\int_0^1 (1-x^2)^{5/2}\, dx$$

$$= \frac{8}{15}\int_{x=0}^{1} (1-x^2)^{5/2}\, dx$$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

When $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow 1 = \sin\theta$

$\theta = \sin^{-1} 1 = \pi/2$

$$= \frac{8}{15}\int_{\theta=0}^{\pi/2} (1-\sin^2\theta)^{5/2} \cos\theta\, d\theta$$

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$$= \frac{8}{15}\int_{\theta=0}^{\pi/2} (\cos^2\theta)^{5/2} \cos\theta \, d\theta = \frac{8}{15}\int_{\theta=0}^{\pi/2} \cos^5\theta \cos\theta \, d\theta$$

$$= \frac{8}{15}\int_{\theta=0}^{\pi/2} \cos^6\theta \, d\theta$$

But $\int_{\theta=0}^{\pi/2} \cos^m\theta \, d\theta = \frac{(m-1)}{m} \times \frac{m-3}{(m-2)} \cdots \pi/2$ if m is even.

$$\therefore \int_{\theta=0}^{\pi/2} \cos^6\theta \, d\theta = \frac{(6-1)}{6} \times \frac{(6-3)}{(6-2)} \times \frac{(6-5)}{(6-4)} \times \frac{\pi}{2} = \frac{5}{\cancel{6}_2} \times \frac{\cancel{3}}{4} \times \frac{1}{2} \times \pi/2$$

$$= \frac{5\pi}{32}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \frac{8}{15}\int_{\theta=0}^{\pi/2} \cos^6\theta \, d\theta = \frac{8}{\cancel{15}_3}\left(\frac{\cancel{5}\pi}{32}\right) = \frac{\pi}{12} \; //.$$

(3) Using Gauss divergence theorem for

If $\vec{F} = (x^2+y)i + (y^2+z)j + (z^2+x)k$ where S is the surface of cube bounded by $x=0, y=0, z=0$ & $x=y=z=1$

Ans: 3 //

(4) Using Gauss divergence theorem for

$\vec{F} = \vec{F} = 4zxi - y^2j + yzk$ where S is the surface of cube bounded by $x=0, x=1, y=0, y=1, z=0$ & $z=1$.

Ans: $3/2$

(5) Using Gauss divergence theorem for

$\vec{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ where S is the surface of rectangular parallelopiped given by $0 \le x, y, z \le a$.

Ans: $abc(a+b+c)$

(6) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where S is enclosed by $0 \le x, y, z \le 1$ & $\vec{F} = x^2i + y^2j + z^2k$

Ans: 3

7) By using Gauss divergence Theorem evaluate $\iint_S \vec{F}\,\hat{n}\,ds$ where $\vec{F} = yi + 2xj - zk$ and S in the Surface of the plane $2x+y=6$ in the first octant Cut off by the plane $z=4$.

Sol: let $\vec{F} = yi + 2xj - zk$

$$\operatorname{div}\vec{F} = \nabla\cdot F = \frac{\partial(y)}{\partial x} + \frac{\partial(2x)}{\partial y} + \frac{\partial(-z)}{\partial z} = -1.$$

let $2x+y=6$. & $z=4$.

for $y=0,\ x\neq 0,\quad 2x=6 \Rightarrow x=3.$

$y\neq 0,\ x\neq 0,\quad 2x+y=6 \Rightarrow y=6-2x.$

$\therefore\ 0\le z\le 4,\ 0\le x\le 3,\ 0\le y\le (6-2x)$

By Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$$= \int_{x=0}^{3}\int_{z=0}^{4}\int_{y=0}^{(6-2x)} (-1)\,dy\,dz\,dx$$

$$= \int_{x=0}^{3}\int_{z=0}^{4} (y)\Big|_{y=0}^{(6-2x)}\,dz\,dx$$

$$= \int_{x=0}^{3}\int_{z=0}^{4} (6-2x)\,dz\,dx$$

$$= \int_{x=0}^{3} (6-2x)(z)\Big|_{z=0}^{4}\,dx$$

$$= \int_{x=0}^{3} 4(6-2x)\,dx$$

$$= \left(4(6x-x^2)\right)\Big|_{x=0}^{3}$$

$$= 4(6(3)-3^2) = 4(18-9) = 4(9) = 36$$

Stokes theorem

Statement: If S be a Surface bounded by a Smooth Closed Curve C and $\vec{F}(x,y,z)$ be a Continuous and differentiable Vector point function on S. Then

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \text{Curl}\,\vec{F} \cdot \hat{n}\, ds$$

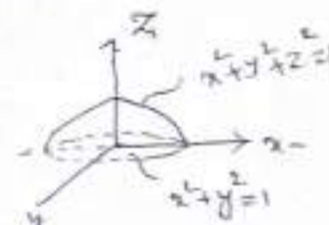
Where $\vec{F} = f_1 i + f_2 j + f_3 k$ & $\hat{n}$ is a Unit Vector at S.

Problems: ① Evaluate Using Stoke's theorem for if $\vec{F} = (2x-y)i + yz^2 j - y^2 z k$ where S is the upper half of the Surface Of the Sphere $x^2+y^2+z^2=1$.

Sol: Let $\vec{F} = (2x-y)i + yz^2 j - y^2 z k$.

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = i(0)+j(0)+k(1) = k$$

$\text{Curl}\,\vec{F} = k$.



By Stoke's theorem we have

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \text{Curl}\,\vec{F}\, \hat{n}\, ds$$

$$= \iint_S (\hat{k} \cdot \hat{k})\, ds = \iint_S ds$$

$$= \text{Surface area of the } x^2+y^2=1$$

$$\oint_C \vec{F}\, d\vec{r} = \pi$$

$\because$ Area of a Circle $x^2+y^2=r^2$ is πr^2, $r=1$

② Evaluate Using Stoke's theorem for If $\vec{F} = yi + zj + xk$ where S is the upper half of the Surface of the Sphere $x^2+y^2+z^2=1$.

Sol. By Stoke's theorem $\oint_C \vec{F}\, d\vec{r} = \iint_S \text{Curl}\,\vec{F} \cdot \hat{n}\, ds$

Let $\vec{F} = yi + zj + xk$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(i+j+k)$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\, ds$$

$$= \iint_S -(i+j+k)\cdot k\, ds.$$

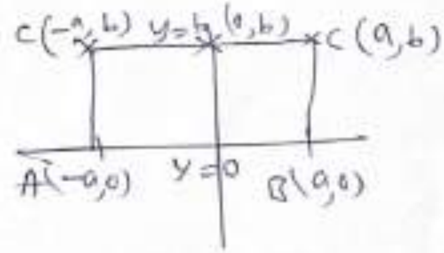
$$= -\iint_S (k\cdot k)\, ds = -\iint_S ds.$$

$= -$(Surface area of the circle $x^2+y^2=1$)

$$= -\pi$$

(3) Using Stoke's theorem evaluate for $\vec{F} = (x^2+y^2)i - 2xyj$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$

Sol: By Stoke's theorem



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\, ds$$

$\vec{F} = (x^2+y^2)i - 2xyj$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4yk$$

$$\text{Curl}\,\vec{F}\cdot\hat{n} = (-4yk)\cdot\hat{k} = -4y$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\,\hat{n}\, ds = \iint_S -4y\, ds$$

$$= \int\int -4y\, dx\, dy$$

$$= \int_{x=-a}^{a}\int_{y=0}^{b} -4y\, dx\, dy = -4\int_{y=0}^{b}\int_{x=-a}^{a} dx\, y\, dy$$

$$= -4ab^2$$

4) Using Stokes theorem evaluate $\oint_C \vec{F} \cdot d\vec{r}$

If $\vec{F} = 2y\,i + 3x\,j - z^2 k$ where C is the boundary of the upper half of the surface of the sphere $x^2+y^2+z^2=9$

Ans: 9π

(Surface area of the circle $x^2+y^2=9$ is 9π)

—— * ——

II BSc - Paper - 4.1

NORMAL SUBGROUP

Kernel of a Homomorphism:

Let $f: G \to G'$ be a homomorphism from the group G into G' and e' be the identity element of G'.
The subset K of G defined by
$K = \{a \in G \mid f(a) = e'\}$ is called the kernel of a Homomorphism f and is denoted by $\ker f$.

Note:
1. Since $f(e) = e'$ $\ker f$ is non empty
2. $x \in \ker f \Rightarrow x \in G$ & $f(x) = e'$
3. If $f(y) = e'$ & $y \in G \Rightarrow y \in \ker f$
4. clearly $\ker f \subseteq G$

Eg:- 1. Let $f: (R^+, \cdot) \to (R, +)$ defined by $f(x) = \log_{10} x$ $\forall x \in R^+$

WKT f is a homomorphism

$\therefore \ker f = \{x \in R^+ \mid f(x) = 0\}$

$= \{x \in R^+ \mid \log_{10} x = 0\}$

$= \{x \in R^+ \mid x = 10^0\}$
$x = 1$

$= \{1\}$

2. Let $f: (R, +) \to (R^+, \cdot)$ be a homomorphism defined by $f(x) = 2^x$ $\forall x \in R$

$\ker f = \{x \in R \mid f(x) = 1\}$

$= \{x \in R \mid 2^x = 1\}$

$= \{x \in R \mid x = 0\}$

$= \{0\}$

3. Let $f:(G,\cdot) \to (G,*)$ be a homomorphism defined by $f(x) = e' \ \forall x \in G$

$\ker f = \{x \in G \mid f(x) = e'\}$

$= G$

Thm: Let $f: G \to G'$ be a homomorphism from G to G' with kernel K then K is a normal subgroup of G

Proof: Given $f: G \to G'$ is a homomorphism

& $K = \{x \in G \mid f(x) = e'$, identity element of $G'\}$ is its kernel

we shall S.T $K \trianglelefteq G$

i. First we P.T $K < G$

Since $f(e) = e'$, $e \in K$

$\therefore K (\neq \phi) \subset G$

Let $x, y \in K$ be arbitrary

$\Rightarrow f(x) = e'$ & $x \in G$

& $f(y) = e'$ & $y \in G$

clearly $xy^{-1} \in G$

Consider $f(xy^{-1}) = f(x) \cdot f(y^{-1})$

$= f(x) \cdot [f(y)]^{-1}$ $\quad \because f$ is a homomor-phism

$= e' \cdot (e')^{-1}$

$= e' \cdot e'$

$= e'$

$\therefore xy^{-1} \in K \quad \forall x, y \in K$

$\therefore K < G$

Let $g \in G$ & $k \in K$ be arbitrary

$\Rightarrow k \in G$ & $f(k) = e'$

We shall P.T $gkg^{-1} \in K$

clearly $gkg^{-1} \in G$

Consider $f(gkg^{-1}) = f(g) \cdot f(k) \cdot f(g^{-1})$ [$\because$ f is homomorphism]

$= f(g) \cdot e' \cdot [f(g)]^{-1}$

$= f(g) \cdot [f(g)]^{-1}$

$= e'$

$\therefore gkg^{-1} \in K \;\; \forall g \in G$ & $k \in K$

$\therefore K \trianglelefteq G$

Theorem:- Let $f: G \to G'$ be a homomorphism with 'K' as its kernel then 'f' is one-one iff $K = \{e\}$ (or)

Let $f: G \to G'$ be a homomorphism with 'K' is kernel then 'f' is an isomorphism $\iff K = \{e\}$

Proof:- Given, $f: G \to G'$ is a homomorphism with 'K' as kernel then 'f' is an isomorphism

Let 'f' be one-one

By defn $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

Let $x \in$ kernel K be arbitrary element.

$\Rightarrow f(x) = e' \to$ identity element of G'

$\Rightarrow f(x) = f(e)$

By the defn of one-one $\quad \because f$ is homomor-phism $f(e) =$

$x = e$

$\therefore K = \{e\}$

Conversely let $K = \{e\}$ we shall S.T 'f' is

Let $f(a) = f(b)$ where $a, b \in G$

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(5)

$\Rightarrow f(a).[f(b)]^{-1} = f(b)[f(b)]^{-1}$
$= e'$

$\Rightarrow f(a.b^{-1}) = e'$

$\Rightarrow a.b^{-1} \in K$

$\Rightarrow a.b^{-1} = e$

$\Rightarrow a(b^{-1}.b) = e.b$

$\Rightarrow a = b$

$\therefore f$ is one-one function.

Thm:- Let G be a group & 'H' be a any normal subgroup of 'G' then the quotient ~~group~~ ~~set~~ group G/H is the homomorphic image of 'G' with 'H' as its ~~~~ kernel.

Proof:- Given 'G' is a group & $H \triangle G$. then G/H exist. Let us define a function $\phi : G \rightarrow G/H$ by

$\phi(x) = Hx \quad \forall x \in G$

clearly ϕ is well defined & onto

Let $a, b \in G$ be arbitrary

$\Rightarrow a, b \in G$

$\phi(a) = Ha \quad \phi(b) = Hb$

$\phi(ab) = Hab$

$\phi(ab) = Ha . Hb$

$= \phi(a) . \phi(b)$

$\therefore \phi$ is a homomorphic

$\ker \phi = \{x \in G \mid \phi(x) = H\}$

$= \{x \in G \mid Hx = H\}$

$= \{x \in G \mid x \in H\}$ [Since $Hx = H \Leftrightarrow x \in H$]

$= H$

$\therefore G/H$ is a homomorphic image of 'G' with H as its kernel

(6)

*** Note: Let $f: G \to G'$ be an onto homomorphism with K as its kernel then f is an isomorphism iff $K = \{e\}$

Problems

(1) Let $G = \{a + b\sqrt{2} \mid a, b \in Q\}$ S.T $T: (G,+) \to (G,+)$ defined by $T(a+b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism

Soln: Let $x, y \in G$ be arbitrary

$x + y \in G$

$x = a_1 + b_1\sqrt{2}$ $y = a_2 + b_2\sqrt{2}$ $\forall a_1, a_2, b_1, b_2 \in Q$

$x + y = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$

$T(x) = a_1 - b_1\sqrt{2}$ $T(y) = a_2 - b_2\sqrt{2}$

$T(x+y) = (a_1 + a_2) - (b_1 + b_2)\sqrt{2}$

$= (a_1 - b_1\sqrt{2}) + (a_2 - b_2\sqrt{2})$

$= T(x) + T(y)$

$\therefore T$ is a homomorphism.

Let $x_1 + y_1\sqrt{2} \in G$ (C.D) be arbitrary

Then $\exists$ $x_1 - y_1\sqrt{2} \in G$ s.t

$T(x_1 - y_1\sqrt{2}) = x_1 + y_1\sqrt{2}$

$\therefore T$ is onto

$\ker K = \{a + b\sqrt{2} \in G \mid T(a + b\sqrt{2}) = 0 + 0\sqrt{2}\}$

$= \{a + b\sqrt{2} \in G \mid a - b\sqrt{2} = 0 - 0\sqrt{2}$

$\Rightarrow a = 0, b = 0 \}$

$= \{0 + 0\sqrt{2}\}$

$= \{0\}$ $\therefore T$ is one-one

$\Rightarrow T$ is an isomorphism

⑦

⑧ Let $f : (C^*, \cdot) \longrightarrow (C^*, \cdot)$ defined by $f(a+ib) = a-ib$ $\forall\, a+ib \in C^*$ where C^* is the set of all non-zero Complex no. Verify f is an isomorphism

Soln:- Let $x, y \in C^*$ be arbitrary

$\Rightarrow x \cdot y \in C^*$

$x = a_1 + ib_1, \quad y = a_2 + ib_2 \quad \forall\, a_1, a_2, b_1, b_2 \in C^*$

$x \cdot y = (a_1+ib_1) \cdot (a_2+ib_2)$

$= a_1a_2 + ia_1b_2 + ib_1a_2 - b_1b_2$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

$\therefore \quad f(x) = f(a_1 + ib_1)$

$= a_1 - ib_1$

$f(y) = f(a_2 + ib_2)$

$= a_2 - ib_2$

$f(x \cdot y) = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1)$ — ①

Consider $f(x) \cdot f(y) = (a_1 - ib_1) \cdot (a_2 - ib_2)$

$= a_1a_2 - ia_1b_2 - ib_1a_2 - b_1b_2$

$= (a_1a_2 - b_1b_2) - (a_1b_2 + b_1a_2)$ — ②

from ① & ② f is an homomorphism.

Let $a+ib \in C^*$ (c.d) be arbitrary

then $\exists\; a-ib \in C^*$ (Domain) s.t

$f(a-ib) = a+ib$

$\therefore f$ is onto

$\ker k = \{a+ib \in C^* \mid f(a+ib) = 1-i(0)\}$ (8)

$= \{a+ib \in C^* \mid a-ib = 1-i(0)$

$\Rightarrow a=1 \quad b=0$

$= \{1\}$

$\therefore f$ is one-one

$\therefore f$ is an isomorphism

(3) Let $f:(R^+,\cdot) \to (R,+)$ be defined by $f(x) = \log_{10} x$ $\forall x \in R^+$

Verify where f is an isomorphism

<u>To S.T f is onto</u>

Let $y \in R(c.d)$ be arbitrary

Let $x \in R^+$ be the preimage of y

$f(x) = y$

$\log_{10} x = y$

$x = 10^y \in R^+$

$\therefore f$ is onto

Let $x, y \in R^+$ be arbitrary

$f(x) = \log_{10} x \quad f(y) = \log_{10} y$

$f(x.y) = \log_{10}(xy)$

$= \log_{10} x + \log_{40} y$ —(1)

$f(x) + f(y) = \log_{10} x + \log_{40} y$ —(2)

from (1) & (2) $f(x.y) = f(x) + f(y)$

$\therefore f$ is an homomorphism

④

$\ker f = \{x \in R^+ / f(x) = 0\}$

$= \{x \in R^+ / \log_{10} x = 0\}$

$= \{x \in R^+ / 10^0 = x\}$ $\Rightarrow x = 1$ $\therefore f$ is one-one

$= \{1\}$ $\therefore f$ is isomorphism.

ii] $f : (R, +) \longrightarrow (C^*, \cdot)$ defined by $f(x) = e^{ix}$ or $f(x) = \cos x + i \sin x$ $\forall x \in R$ where

Let $a, b \in R$ be any 2 arbitrary

$\Rightarrow a+b \in R$

We have $f(a) = e^{ia}$ $f(b) = e^{ib}$

$f(a+b) = e^{i(a+b)}$ — ①

$f(a) \cdot f(b) = e^{ia} \cdot e^{ib}$

$= e^{i(a+b)}$ — ②

from ① & ②

$f(a+b) = f(a) \cdot f(b)$

$\therefore f$ is homomorphism

(or)

$f(a) = \cos a + i \sin a$

$f(b) = \cos b + i \sin b$

$f(a+b) = \cos a + i \sin a + \cos b + i \sin b$

$= (\cos a + \cos b) + i(\sin a + \sin b)$

$f(a) \cdot f(b) = (\cos a + i \sin a) \times (\cos b + i \sin b)$

$= (\cos a \cdot \cos b) + i(\sin a \cdot \sin b)$

$\ker f = \{x \in R / f(x) = e^{i(0)}\}$

$= \{x \in R / e^{ix} = e^{i(0)}\}$

$= \{0\}$

To S.T f is onto

Let $y \in C^*$ be arbitrary

Let $x \in R$ be the pre image of y

$f(x) = y$

$e^{ix} = y$

$\log y = x \in R$

$\therefore f$ is onto

5] Let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} / a \in R^* \right\}$ & $f : (G, \cdot) \longrightarrow (R^*, \cdot)$ be defined by $f\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right] = a \;\forall \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in G$ (10)

Let $x, y \in G$ be arbitrary

we have $f\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = x \quad f\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = y$

$f\begin{pmatrix} xy & 0 \\ 0 & 0 \end{pmatrix} = xy$ — ①

$f\left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \cdot f\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}\right] = x \cdot y$ — ②

from ① & ② =

f is homomorphism

$\text{Ker} k = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in G / f\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$x = 1$

$= \{1\}$

Let $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in R^*$ be arb

Let $\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \in G$ be preimage of $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$

$f\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$

$y = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in R^*$

f is on-to

** Let $\phi : (Z, +) \to (2Z, +)$ be arbitrary $\phi(x) = 2x, \forall x \in Z$

Let $x, y \in Z$ be arbitrary

$\phi \Rightarrow x + y \in Z$

$\phi(x) = 2x \quad \phi(y) = 2y$

$\phi(x+y) = 2(x+y)$ — ①

$\phi(x) + \phi(y) = 2x + 2y = 2(x+y)$ — ② $\forall x, y \in Z$

from ① & ②

ϕ is homomorphism

Ker $K = \{x \in Z \mid \phi(x) = 0\}$

$= \{x \in Z \mid 2x = 0\}$

$= \{0\}$

$\therefore \phi$ is one-one

Let $2n \in 2Z$ be arbitrary

$\Rightarrow n \in Z$

$\Rightarrow \phi(n) = 2n$

$\therefore \phi$ is on-to

$\therefore \phi$ is an isomorphism

*** Fundamental theorem of homomorphism (FTH)

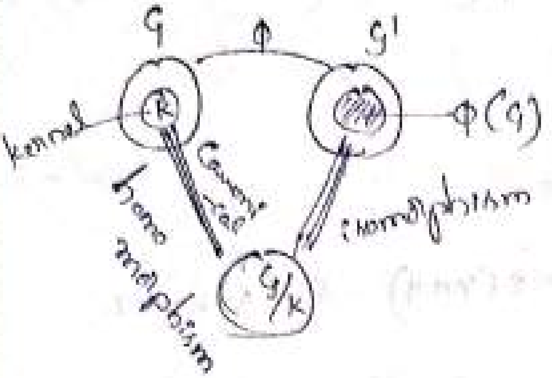
Statement: Every homomorphic image of group is isomorphic to its quotient groups there of

(or)

Let $\phi : G \to G'$ be a homomorphism with kernel K. Then $G/K \cong \phi(G)$

(or)

Let ϕ be a homomorphism from G onto G' with kernel K then $G/_K \cong G'$



Proof: Given $\phi : G \to G'$ is a homomorphism with kernel K

i.e. $K = \{ x \in G \mid f(x) = e' \text{ identity element of } G' \}$

Since $K \triangle G$

$\therefore G/_K$ exist

$G/_K = \{ Ka / a \in G \}$

& $\phi(G)$ is the homomorphic image of G under ϕ

$\phi(G) = \{ \phi(x) / x \in G \}$

Let us define a function $\varphi : G/K \to \phi(G)$ by

$\varphi(Ka) = \phi(a) \quad \forall \, Ka \in G/_K$

We shall S.T φ is ~~hom~~ Isomorphism

We show φ is well defined and one-one

Let $Ka = Kb$

$\Leftrightarrow ab^{-1} \in K \quad (\because K \triangle G)$

$\Leftrightarrow \phi(ab^{-1}) = e'$

$\Leftrightarrow \phi(a)\, \phi(b^{-1}) = e' \quad \because \phi$ is homomorphism

$H \triangle G$
$Ha = Hb$
$ab^{-1} \in H$

$\Leftrightarrow \phi(a)[\phi(b)]^{-1} = e'$

$\Leftrightarrow \phi(a)[\phi(b)]^{-1}\phi(b) = e'\phi(b)$

$\Leftrightarrow \phi(a).e' = \phi(b)$

$\Leftrightarrow \phi(a) = \phi(b)$

$\Leftrightarrow \psi(ka) = \psi(kb)$

13

$\therefore$ ψ is well defined & one-one

Let $kx, ky \in G/K$ be arbitrary

$\psi(kx) = \phi(x) \quad \psi(ky) = \phi(y)$

now $kx.ky = kxy \in G/K$

Now $\psi(kx.ky) = \psi(kxy) = \phi(x.y)$

$= \phi(x)\phi(y)$ ($\because \phi$ is homo)

$= \psi(kx)\psi(ky)$

$\psi(kx.ky) = \psi(kx)\psi(ky)$

$\forall$ any $kx, ky \in G/K$

$\therefore$ ψ is a homomorphism

<u>To show ψ is onto</u>

Let $y \in \phi(G)$ be arbitrary

then $\exists$ $x \in G$ s.t $\phi(x) = y$

Now for $x \in G$, $\exists$ $kx \in G/K$

$\ni$: $\psi(kx) = \phi(x) = y$

$\therefore$ kx is the preimage of y

$\therefore$ ψ is onto

$\therefore$ ψ is an isomorphism $\quad \therefore G/K \cong \phi(G)$

Some Results on Isomorphism

Statement :- If $f : G \to G'$ be an isomorphism of a group G onto a group G' and a is any element of G, then the order of $f(a)$ equals the order of a.

Proof :- Let e and e' be the identity elements of G and G'

Case(i) :- Let a be of finite order, say n.

$\therefore$ n is the least positive integer such that $a^n = e$

Now, $a^n = e$ $\Rightarrow f(a^n) = f(e)$

$\Rightarrow [f(a)]^n = e'$ [since f is homomorphism]

$\Rightarrow f(a)$ is also of finite order.

Let $O[f(a)] = m$ i.e., m is the least positive integer such that

$[f(a)]^m = e'$

Also we have $[f(a)]^n = e'$. $\therefore$ m divides n.

Now, $[f(a)]^m = e'$ $\Rightarrow f(a^m) = f(e)$

$\Rightarrow a^m = e$. [since f is one-one]

But $o(a) = n$ Thus n divides m.

$\therefore$ m divides n and n divides m.

~~Group Theory~~ Hence $m = n$ i.e $O[f(a)] = O(a)$.

Case(ii) :- Let a be of infinite order

Let $f(a)$ be finite order say m.

That m is the least positive integer such that $[f(a)]^m = e'$.

$\therefore$ $f(a^m) = e'$

$\Rightarrow$ $f(a^m) = f(e) \Rightarrow (a)^m = e$

$\therefore$ a is of finite order. This is a contradiction.

$\therefore$ $O[f(a)]$ is Infinite

2] Let $f: G \to G'$ be an isomorphism. If G is abelian, then G' is also abelian.

Proof :- Let $a', b' \in G'$ be arbitary

Since f is onto, $\exists\, a, b \in G$ such that
$$f(a) = a', \; f(b) = b'$$

Consider
$$\begin{aligned} a'b' &= f(a) \cdot f(b) \\ &= f(ab) = f(ba) \quad [\text{Since } G \text{ is abelian}] \\ &= f(b) \cdot f(a) = b' \cdot a' \end{aligned}$$

$\therefore \quad a'b' = b'a' \; \forall\, a', b' \in G'$.

Hence G' is abelian.

3> Any infinite Cyclic group is isomorphic to the group z of integers, under addition.

Proof :- Let G be an infinite cyclic group having generator 'a'.

$\therefore$, $\quad G = \{a^n : n \in z\}$

Define $\quad f: G \to z$

by $\quad f(a^n) = n \; \forall\, a^n \in G$

i> f is homomorphism.

Consider, $\quad f(a^m \cdot a^n) = f(a^{m+n})$
$$= m+n = f(a^m) + f(a^n)$$

$\therefore \quad f(a^m \cdot a^n) = f(a^m) + f(a^n)$

$\therefore$ f is homomorphism

ii) f is onto

$\forall n \in Z$, $a^n \in G$, G being infinite group

$\therefore \forall n \in Z, \exists a^n \in G$ Such that $f(a^n) = n$.

$\therefore f$ is onto

iii) f is one-one

Let $f(a^n) = f(a^m) \Rightarrow n = m \Rightarrow a^n = a^m$

$\therefore f$ is one-one.

Hence f is isomorphism

Permutation group

Permutation group is an ~~one-one~~ & onto (bijective) function from a finite set onto itself.

The no of elements in finite set is c/d degree of permutation. A symmetric group S_n the set of all permutation defined on set S containing n elements forms a group under the composition of product of permutations. This group denoted by S_n is c/d symmetric group of degree 'n' & order n!

i) S_3 is the symmetric group on 3 symbols say {1, 2, 3}

ie $S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right.$

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \}$

note:- (iii) $\forall n \geq 3$, S_n is a non-abelian group

① Find the combination of the permutations $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ & $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ defined on $S = \{1, 2, 3, 4\}$.

$$f.g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 2 & 4 \end{matrix}$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 \end{pmatrix}$

② Find the inverse of the permutation $f = \begin{pmatrix} a & b & c & d & e \\ e & b & c & d & a \end{pmatrix}$ defined on $S = \{a, b, c, d, e\}$

$$f^{-1} = \begin{pmatrix} a & b & c & d & e \\ e & b & c & d & a \end{pmatrix}$$

* Any subgroup of S_n is cld permutation group of degree n.

Cayley's Theorem

Statement:- Every finite group is isomorphic to a permutation group (or)

Every finite group is isomorphic to a subgroup of Symmetric group.

Proof:- Let 'G' be a finite group of ~~order~~ order 'n'.

$\forall g \in G$, Let us define a function $T_g : G \to G$ by

$$T_g(x) = xg \quad \forall x \in G$$

T_g is 1-1 ; $T_g(x_1) = T_g(x_2)$

$$\Rightarrow x_1 g = x_2 g$$

$\Rightarrow x_1 = x_2$

T_g on to; Let $y \in G$ be arbitrary such that

$T_g(yg^{-1}) = (yg^{-1})g = y$ (17)

Then $\exists\, yg^{-1} \in G$ (Domine)

$\therefore\ \forall g \in G$, T_g is a permutation from G onto itself

Let $G' = \{ T_g \mid g \in G \}$

clearly $G' (\neq \phi) \subset S_n$

We shall P.T $G' < S_n$

Let $T_{g_1}, T_{g_2} \in G'$ be arbitrary

Consider $(T_{g_1} \cdot T_{g_2})(x) = T_{g_2}(T_{g_1}(x))$

$= T_{g_2}(xg_1)$

$= xg_1g_2$

$T_{g_1g_2}(x) = x \cdot g_1 g_2$

$\therefore\ \boxed{T_{g_1} \cdot T_{g_2} = T_{g_1g_2}} \in G'$

Consider $T_g \cdot T_{g^{-1}}(x) = T_{g^{-1}}(xg)$

$= xgg^{-1}$

$= x\ \forall x \in G$

$\therefore\ T_g \cdot T_{g^{-1}} = T_e$

$\therefore\ T_{g^{-1}}$ is the inverse of $T_g\ \forall g \in G$

$\therefore\ (T_g)^{-1} = T_{g^{-1}} \in G'$

$\therefore\ G' < S_n$

Let us define $\phi : G \to G'$ by $\phi(g) = T_g\ \forall g \in G$

To show that ϕ is 1-1

$\phi(g_1) = \phi(g_2)$

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$\tau_{g_1} = \tau_{g_2}$

$\tau_{g_1}(x) = \tau_{g_2}(x) \ \forall x \in G$

$xg_1 = xg_2$

$\Rightarrow g_1 = g_2$

$\therefore \phi$ is 1-1

To S.T ϕ is onto:

Let $\tau_y \in G'$ be arbitrary

$\Rightarrow y \in G$

$\Rightarrow \phi(y) = \tau_y$

$\therefore \phi$ is onto

Let $g_1, g_2 \in G$ be arbitrary

$g_1 g_2 \in G$

$\phi(g_1) = \tau_{g_1}$

$\phi(g_2) = \tau_{g_2}$

$\phi(g_1 \cdot g_2) = \tau_{g_1 g_2}$

$= \tau_{g_1} \cdot \tau_{g_2}$

$= \phi(g_1)\, \phi(g_2)$

$\therefore \phi$ is homomorphism

$\therefore \phi$ is Isomorphism

$\therefore G \cong G'$

$\therefore$

Note:- The set G' in the above theorem is cld right regular representation of G

① Find the right regular representation of the group $G = \{1, -1, i, -i\}$

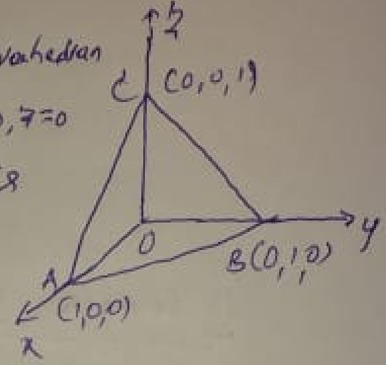
IV semester II BSc

H.R part remaining syllabus notes

Triple integral

1. Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the region bounded by the coordinate planes and the planes $x+y+z=1$

Soln The region of integration is tetrahedran bounded by the planes $x=0, y=0, z=0$ and the plane $x+y+z=1$ which is shown in the figure.



clearly, 'z' varies from $z=0$ to $z=1-x-y$. for $x=0$

y varies from $y=0$ to $y=1-x$ Now for $y=0, z=0$, the 'x' varies from $x=0$ to $x=1$.

$$\therefore \iiint \frac{dx\,dy\,dz}{(x+y+z)^3} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left.\frac{(x+y+z+1)^{-2}}{-2}\right|_{z=0}^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} 2^{-2} - (x+y+1)^{-2}\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left. 2^{-2}\cdot y - \frac{(x+y+1)^{-1}}{-1}\right|_{y=0}^{1-x} dx$$

by

$$= -\frac{1}{2}\int_0^1 \frac{1-x}{4} + 2^{-1} - \frac{1}{x+1}\, dx$$

$$= -\frac{1}{2}\left[\frac{(1-x)^2}{4(-2)} + \frac{x}{2} - \log(x+1)\right]_{x=0}^{1}$$

$$= -\frac{1}{2}\left[0 + \frac{1}{2} - \log 2 - \frac{1}{-8} - 0 + 0\right]$$

$$= \frac{\log 2}{2} - \frac{1}{4} - \frac{1}{16} = \frac{\log 2}{2} - \frac{5}{16} = \underline{\log\sqrt{2} - \frac{5}{16}}$$

2. Find the volume of the tetrahedron formed the planes $x=y=z=0$ and $6x+4y+3z=12$.

Sol: The volume of the region 'V' in the space is given by

$$V = \iiint_V dx\,dy\,dz$$

Now, to cover the region 'V' 'z' varies from $z=0$ to $z=\frac{1}{3}(12-6x-4y)$. 'y' varies from $y=0$ to $y=\frac{1}{4}(12-6x)$ ($\because z=$ and 'x' varies from $x=0$ to $x=2$ ($\because y=z=0.$)

$$\therefore V = \iiint_V dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{\frac{1}{4}(12-6x)}\int_{z=0}^{\frac{1}{3}(12-6x-4y)} dz\,dy\,dx$$

$$= \int_0^2\int_0^{\frac{1}{4}(12-6x)} \frac{1}{3}(12-6x-4y)\,dy\,dx$$

$$= \frac{1}{3}\int_0^2 \frac{(12-6x-4y)^2}{2(-4)}\Bigg|_{y=0}^{\frac{1}{4}(12-6x)} dx$$

$$= -\frac{1}{24}\int_0^2 \left[\left(12-6x-4\cdot\frac{1}{4}(12-6x)\right)^2 - (12-6x)^2\right] dx$$

$$= -\frac{1}{24}\int_0^2 0-(12-6x)^2 dx$$

$$= \frac{1}{24}\left[\frac{(12-6x)^3}{3(-6)}\right]_0^2 = \frac{1}{24}\left(0-\frac{12^3}{-18}\right)$$

$$= \frac{12\cdot 12\cdot 12}{12\cdot 2\cdot 18} = 4$$

$\therefore$ Volume of the tetrahedron is $\underline{\underline{4}}$

3. Find the volume bounded by the surface $z = a^2 - x^2$ and the plane $x=0, y=0, z=0$ and $y=b$.

Sol^n: To cover the volume 'V'. Here 'z' varies from $z=0$ to $z=a^2-x^2$. for $z=0$, 'y' varies from $y=0, y=b$ and for $y=0, z=0$ 'x' varies from $x=0, x=a$.

$$\therefore \text{Volume} = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{a^2-x^2} dz\,dy\,dx$$

$$= \int_0^a\int_0^b z\Big|_0^{a^2-x^2} dy\,dx$$

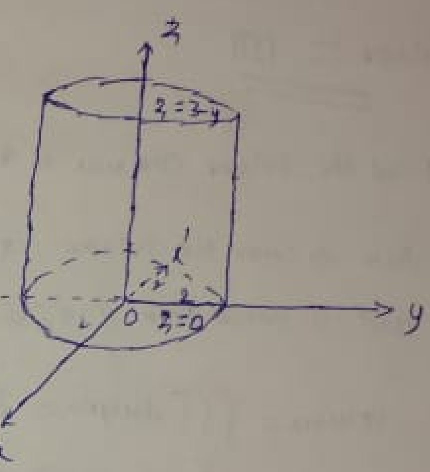
$$= \int_0^a\int_0^b a^2-x^2\,dy\,dx$$

$$= a^2x - \frac{x^3}{3}\Big|_0^a \; y\Big|_0^b = \left(a^3 - \frac{a^3}{3}\right)\times(b-0) = \frac{2a^3}{3}\times b$$

$\therefore$ Volume $= \dfrac{2a^3b}{3}$

4. Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=3$ and $z=0$.

Soln: By data, to cover the volume 'v', 'z' varies from $z=0$ to $z=3-y$, 'y' varies from $y=-\sqrt{4-x^2}$ to $y=\sqrt{4-x^2}$ and 'x' varies from $x=-2$ to 2



$$\text{volume} = \iiint_V dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3-y} dz\,dy\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z\Big|_{z=0}^{3-y} dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-y)\,dy\,dx$$

$$= \int_{-2}^{2} 3y - \frac{y^2}{2}\Big|_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[3\sqrt{4-x^2} - \frac{4-x^2}{2} + 3\sqrt{4-x^2} + \frac{4-x^2}{2}\right]dx$$

$$= 6\int_{-2}^{2}\sqrt{4-x^2}\,dx = 12\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 12\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 12\left[0 + \frac{4}{2}\cdot\frac{\pi}{2} - 0\right]$$

Volume $= 12\pi$

5. Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Soln: Here to cover the volume, 'z' varies from $z=\pm\sqrt{a^2-x^2}$ and 'y' varies from $y=\pm\sqrt{a^2-x^2}$ and 'x' varies from $x=\pm a$.

$$\therefore \text{volume} = \iiint_V dz\,dy\,dx = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\,dx$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\cdot 2\sqrt{a^2-x^2}\,dx$$

$$= 4\int_{-a}^{a}(a^2-x^2)\,dx = 8\int_{0}^{a}(a^2-x^2)\,dx$$

$$= 8\left[a^2x - \frac{x^3}{3}\right]_0^a = 8\left(a^3 - \frac{a^3}{3}\right) = 8\cdot\frac{2a^3}{3}$$

$$\therefore \text{Volume} = \frac{16a^3}{3}$$

6. Find the volume of the tetrahedran bounded by the coordinate p... and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol: Here to cover the volume 'V' bounded by $x=0, y=0, z=0$ and th... plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ Thus, 'z' varies from $z=0$ to $z=c(1-\frac{x}{a}$... 'y' varies from $y=0$, to $y=b(1-\frac{x}{a})$ for $z=0$ and 'x' varies from $x=0$ to $x=a$ for $y=0, z=0$

$\therefore$ Volume of the tetrahedran $= \iiint dz\,dy\,dx$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_{0}^{a}\int_{0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_{0}^{a} \frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2\left(-\frac{1}{b}\right)}\Bigg|_{y=0}^{b(1-\frac{x}{a})} dx$$

$$= -\frac{bc}{2}\int_{0}^{a} 0-\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\,\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\Bigg|_{0}^{a} = -\frac{abc}{6}(0-1)$$

$$\text{Volume} = \frac{abc}{6}$$

Change of variables in the triple integral

A triple integral $\iiint_V f(x,y,z)\,dx\,dy\,dz$ can be evaluated by changing the variables x,y,z to new variables u,v,w which are related to x,y,z. In this case if

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0.$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_D \phi(u,v,w)\,|J|\,du\,dv\,dw.$$

We shall take two particular transformation.

(i) Cylindrical polar coordinates

Let (x,y,z) be related to the variables (R,ϕ,z) by $x = R\cos\phi$, $y = R\sin\phi$, $z = z$.

$$\therefore J = \frac{\partial(x,y,z)}{\partial(R,\phi,z)} = R \neq 0 \quad \therefore dx\,dy\,dz = R\,dR\,d\phi\,dz.$$

1. Evaluate $\iiint_R xyz\,dx\,dy\,dz$ over the +ve octant of the sphere $x^2+y^2+z^2=a^2$ by transforming into cylindrical polar coordinates.

Soln. In the given region 'R' of integration, 'R' varies from $R=0$ to $R=a$, ϕ varies from $\phi=0$ to $\phi=\frac{\pi}{2}$ and 'z' varies from $z=0$ to $z=0$ since 'R' is +ve octant for $x=R\cos\phi$, $y=R\sin\phi$, $z=z$, then $dx\,dy\,dz = R\,dR\,d\phi\,dz$.

$$\therefore \iiint_R xyz\,dx\,dy\,dz = \frac{1}{3}\int_{R=0}^{a}\int_{\phi=0}^{\frac{\pi}{2}}\int_{z=0}^{a} R\cos\phi \cdot R\sin\phi \cdot zR\,dz\,d\phi\,dR.$$

z

$$= \frac{1}{3}\int_0^a R^3 dR \times \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \times \int_0^a z\, dz$$

$$= \frac{1}{3}\cdot\frac{R^4}{4}\Big|_0^a \times \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} \times \frac{z^2}{2}\Big|_0^a$$

$$= \frac{1}{3}\cdot\frac{a^4}{4} \times \frac{1}{2} \times \frac{a^2}{2} = \frac{1}{3}\frac{a^6}{16} \therefore = \frac{a^6}{48}.$$

2. Evaluate $\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz \; dz\,dy\,dx$ by changing into cylindri polar coordinates.

Soln $x=0, x=1, y=0, y=1$ & $z=\sqrt{x^2+y^2}, z=2$. then $x=R\cos\phi, y=R\sin\phi, z=$

To cover, $R=0$ to $R=\sqrt{2}$ $\because R^2 = x^2+y^2 \therefore R^2 = 1+1 = 2$.

'ϕ' varies from $\phi=0, \phi=\frac{\pi}{4}$ $\therefore$ $\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

and 'z' varies from $z=1$ to $z=2$.

Consider

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz\, dz\,dy\,dx = \int_{R=0}^{\sqrt{2}}\int_{\phi=0}^{\frac{\pi}{4}}\int_{z=1}^{2} R\cos\phi \cdot R\sin\phi \cdot z \cdot R\,d\phi\,dR.$$

$$= \int_{R=0}^{\sqrt{2}} R^3 dR \cdot \int_{\phi=0}^{\frac{\pi}{4}} \sin\phi\cos\phi\, d\phi \cdot \int_{z=1}^{2} z\, dz$$

$$= \frac{R^4}{4}\Big|_0^{\sqrt{2}} \cdot \frac{\sin^2\phi}{2}\Big|_0^{\frac{\pi}{4}} \cdot \frac{z^2}{2}\Big|_1^2$$

$$= \frac{4}{4} \times \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} \times \frac{4-1}{2} = \frac{1}{4}\times\frac{3}{2}$$

$$= \frac{3}{8} //$$

(ii) Spherical polar coordinates:-

Let (x, y, z) be related to three variables (ρ, θ, ϕ) by the relation $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\theta$.

Then (ρ, θ, ϕ) are called spherical polar coordinates.

In this case $J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\theta$.

Thus, $\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_D \phi(\rho,\theta,\phi)\,d\rho\,d\theta\,d\phi$.

1. Evaluate $\iiint_R xyz\,dx\,dy\,dz$ over the +ve octant of the sphere $x^2+y^2+z^2=a^2$ by changing it to spherical polar coordinates.

Soln In the region 'R', the 'ρ' varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$ and 'ϕ' varies from 0 to $\frac{\pi}{2}$ and for the spherical coordinates $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\theta$. also $dx\,dy\,dz = \rho^2\sin\theta$.

$$\therefore \iiint_R xyz\,dx\,dy\,dz = \int_{\rho=0}^{a}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \rho\sin\theta\cos\phi \cdot \rho\sin\theta\sin\phi \cdot \rho\cos\theta \cdot \rho^2\sin\theta \; d\rho\,d\theta\,d\phi$$

$$= \int_0^a \rho^5\,d\rho \times \int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta \times \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi$$

$$= \left.\frac{\rho^6}{6}\right|_0^a \times \left.\frac{\sin^4\theta}{4}\right|_0^{\pi/2} \times \left.\frac{\sin^2\phi}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{a^6}{6} \times \frac{1}{4} \times \frac{1}{2} = \frac{a^6}{48}.$$

2. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing into the Spherical polar coordinates.

Soln. For the given limits, $x = 0$ to 1, $y = 0$ to $\sqrt{1-x^2}$ & $z = 0$ to $\sqrt{1-x^2-y^2}$

$\therefore z^2 = 1 - x^2 - y^2$ (or) $x^2+y^2+z^2 = 1$. thus, the region of integration is first octant (or) positive octant of the space.

$\therefore x = \rho\sin\theta\cos\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\theta$ & $dx\,dy\,dz = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi$

and ρ varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\rho^2\sin\theta\,d\phi\,d\theta\,d\rho}{\sqrt{1-\rho^2}}$$

$$= \int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}}\,d\rho \times \int_0^{\pi/2} \sin\theta\,d\theta \times \int_0^{\pi/2} d\phi$$

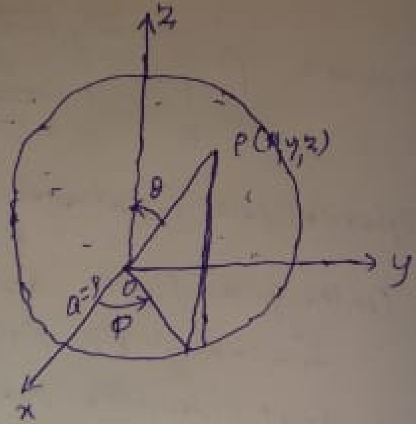
put $\rho = \sin t$.
$d\rho = \cos t\,dt$
$\rho = 0,\ t = 0$
$\rho = 1,\ t = \pi/2$

$$= \int_0^{\pi/2} \sin^2 t\,dt \times (-\cos\theta)\Big|_0^{\frac{\pi}{2}} \times \phi\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} \times 1 \times \frac{\pi}{2} = \frac{\pi^2}{8}.$$

3. Find the volume of the sphere $x^2+y^2+z^2 = a^2$ using triple integral.

Soln. Let 'V' be the volume of the sphere $\therefore V = \iiint_V dx\,dy\,dz$. the region of integration is whole sphere. then 'ρ' varies from 0 to 'a' and 'θ' varies from 0 to π and 'ϕ' varies from 0 to 2π, Also $dx\,dy\,dz = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi$.



∴ Volume of the Sphere

$$V = \iiint_V dx\,dy\,dz$$

$$= \int_{\rho=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho^2 \sin\theta \, d\phi \, d\theta \, d\rho$$

$$= \int_0^a \rho^2 d\rho \times \int_0^\pi \sin\theta \, d\theta \times \int_0^{2\pi} d\phi$$

$$= \frac{\rho^3}{3}\Big|_0^a \times (-\cos\theta)\Big|_0^\pi \times \phi\Big|_0^{2\pi}$$

$$= \frac{a^3}{3} \times (1+1) \times 2\pi = \frac{a^3}{3} \times 4\pi = \frac{4\pi a^3}{3}$$

∴ Volume of the Sphere $= \frac{4\pi a^3}{3}$